Observer-Based Robust Control Giving Consideration to Transient Behavior for Linear Uncertain Discrete-Time Systems

Hidetoshi OYA[†] and Kojiro HAGINO[‡]
Department of Systems Engineering, The University of Electro-Communications [†] [‡]
1-5-1 Chofugaoka Chofu-city Tokyo, JAPAN [†] [‡]
(Tel:+81-424-43-5253,E-mail:hide-o@se.uec.ac.jp [†])
(Tel:+81-424-43-5242,E-mail:hagino@se.uec.ac.jp [‡])

Abstract– In this paper, we present an observer-based robust controller which achieves not only robust stability but also an performance robustness for linear uncertain discrete-time systems. The performance robustness means that comparing the transient behavior of the uncertain system with a desired one generated by the nominal system, the deterioration of control performance (i.e. the error between the real response and the desired one) is suppressed without excessive control input. The control law consists of a state feedback law for the nominal system and a compensation input given by a feedback form of an estimated error signal. In this paper, we show that conditions for the existence of the observer-based controller are given in terms of linear matrix inequalities (LMIs). Finally, a numerical example is given to illustrate the proposed technique.

Key Words: observer-based controller, compensation input, keeping transient behavior, LMIs

1 Introduction

Recently, some researchers investigated quadratic stabilizing control with achievable performance level in reference to such as a quadratic cost function[1, 2] and performance robust H_2 control [3]. On the other hand, in the case that the full state information of the plant cannot be measured, some observer-based robust controllers [4] and output feedback control systems [5] are presented. These methods, however, result in worst-case design, and therefore, these controllers become cautious when the perturbation region of the uncertainties has been estimated larger than the proper region, since the controller designed by these methods only has a fixed gain.

By the way, in most practical situations, design specifications are given for the transient behavior of the system such as rise time, overshoot and so on. Therefore, it is necessary that the deterioration of control performance such as not only the cost performance but also the transient behavior in time response should be suppressed. Now, it is well-known that the desirable transient behavior relating to the damping ratio of the closed-loop response of linear continuous-time systems can often be achieved by forcing the closed-loop poles into suitable subregions of the complex left-half plane. Thus, there have been several researches dealing with the problem of designing a robust controller that satisfies additional constraints on the closed-loop poles' location [6]. However, it has not been established for general linear multi-input multi-output systems in which subregion closed-loop poles should be located in order to achieve a specified transient behavior.

In contrast with these for linear systems with structured uncertainties, we have derived a robust controll scheme with compensation input [7, 8]. In this approach, the control law consists of a state feedback with a fixed gain designed by using the nominal system and a compensation input defined as a feedback law of an error signal or estimated error signal between the response of the uncertain system and the desired one generated by the nominal system. The controller is more flexible and adaptive than conventional robust controllers with a fixed gain, because by utilizing the error signal, the controllers can reflect the effect of uncertainties.

In this paper, we extend the above mentioned robust controller[7, 8] to an observer-based robust controller for linear discrete-time systems with structured uncertainties. The control law consists of a state feedback law designed by using the nominal system and a compensation input for the purpose of reducing the effect of the uncertainties. Since the error between the response of the uncertain system and the desired one generated by the nominal system cannot be used directly, an observer is introduced in order to estimate an error.By using the estimated error signal, the compensation input is given by an estimated error signal feedback. In this paper, firstly, the feedback law for the nominal system is obtained and next, the observer is designed. Finally, by using linear matrix inequalities (LMIs) technique, the compensation input is determined so that an upper bound on a given quadratic cost function for an augmented system consisting of the observer and an estimation error system.

This paper is organized as follows. In Section 2, we introduce the class of uncertain system under consideration, the error system and the observer for the error information together with corresponding quadratic cost function. Section 3 contains the main results. The conditions for the existence of the compensation input to minimize the upper bound on the quadratic cost function for the error system are given in terms of linear matrix inequalities (LMIs). Finally, an simple example is included to illustrate the results developed in this paper.

2 Problem formulation

Consider the following uncertain system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A(\theta)\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C(\theta)\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k) \in \Re^n, \boldsymbol{u}(k) \in \Re^m$ and $\boldsymbol{y}(k) \in \Re^l$ are the vectors of the state, the control input and the measured output, respectively. The matrix $A(\theta)$ is supposed to have appropriate dimension and following structure

$$A(\theta) = A + \sum_{i=1}^{N} \theta_i A_i, \quad C(\theta) = C + \sum_{i=1}^{N} \theta_i C_i \tag{2}$$

In eq.(2), the matrix A denotes the nominal value and the matrices A_i for $i = 1, \cdots, N$ represent the structure of the uncertainties. The unknown parameter vector $\theta \in \Re^N$ $(\theta = (\theta_1, \cdots, \theta_N)^T)$ satisfies the relation $\sum_{i=1}^{N} \theta_i = 1,\ \theta_i \geq 0$ for $i = 1, \cdots, N$. We assume that the set of the $\theta \in \Re^N$ is described as Δ and for $\forall \theta \in \Delta$, the pairs $(A(\theta), B)$ and $(A(\theta), C)$ are controllable and observable.

The nominal system, ignoring the unknown parameters θ_i in eq.(1), is given by

$$\begin{aligned} \boldsymbol{x}_r(k+1) &= A\boldsymbol{x}_r(k) + B\boldsymbol{u}_r(k) \\ \boldsymbol{y}_r(k) &= C\boldsymbol{x}_r(k) \end{aligned} \tag{3}$$

where $\boldsymbol{x}_r(k) \in \Re^n, \boldsymbol{u}_r(k) \in \Re^m$ and $\boldsymbol{y}_r(k) \in \Re^l$ are the vectors of the state, the control input and the measured output, respectively.

In this paper, first of all, we generate a desirable transient behavior in time response for the uncertain system eq.(1) adopting the standard linear quadratic control problem for the nominal system. It is well-known that the optimal control input for the nominal system eq.(4) can be obtained as

$$\overline{\boldsymbol{u}}(k) = -K\overline{\boldsymbol{x}}(k) \tag{4}$$

In (4), the feedback gain matrix K is obtained as $K = (R + B^TPB)^{-1}B^TPA$ by using the solution of the following algebraic Riccati equation.

$$\begin{aligned} P &= A^TPA - P + Q \\ &\quad -A^TPB(R + B^TPB)^{-1}B^TPA \end{aligned} \tag{5}$$

where the weighting matrices $Q \in \Re^{n\times n}$ and $R \in \Re^{m\times m}$ for the quadratic cost function are nonnegative and positive definite, respectively and are determined in advance so that the desirable transient behavior in time response for the uncertain system eq.(1) is achieved. Furthermore, using the matrix H which satisfies detectability of the pair (H, A), the weighting matrix Q is given as $Q = H^TH$. Then, the closed-loop system for the nominal system given by applying the optimal control law eq.(4) is asymptotically stable[9].

For the uncertain system eq.(1), we consider the following control law.

$$\boldsymbol{u}(k) \stackrel{\triangle}{=} \overline{\boldsymbol{u}}(k) + \boldsymbol{v}(k) \tag{6}$$

where $\boldsymbol{v}(k)$ is the compensation input for the purpose of modifying the control input in order to reduce the error between the real trajectory and the desired one[7, 8]. In other words, the compensation input is additional corrective signal in order to keep the satisfactory transient response.

If we define an error vector $\boldsymbol{e}(k) \stackrel{\triangle}{=} \boldsymbol{x}(k) - \overline{\boldsymbol{x}}(k)$, then from eqs.(1),(3),(4) and (6), the following error system is derived

$$\boldsymbol{e}(k) = A(\theta)\boldsymbol{e}(k) + B\boldsymbol{v}(k) + A_e(\theta)\overline{\boldsymbol{x}}(k) \tag{7}$$

where $A_e(\theta)$ is the matrix given by $A_e(\theta) = A(\theta) - A$.

In this paper, we consider to design the compensation input $\hat{\boldsymbol{v}}(k)$ such that the satisfactory transient response as closely as possible to the desirable one is achieved.However by using the error information

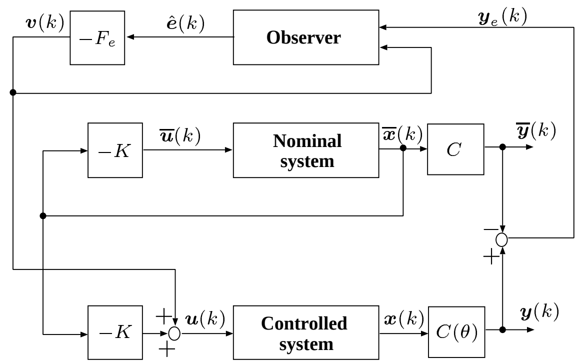

Fig. 1: Configuration of the proposed robust control system

$\hat{\boldsymbol{e}}(k)$, the compensation input $\boldsymbol{v}(k)$ cannot be designed because the error signal cannot be utilized directly. Thus for the error vector $\hat{\boldsymbol{e}}(k)$, we introduce the following full state observer

$$\begin{aligned} \hat{\boldsymbol{e}}(k+1) &= A\hat{\boldsymbol{e}}(k) + B\boldsymbol{v}(k) \\ &\quad + G_e(\boldsymbol{y}_e(k) - C\hat{\boldsymbol{e}}(k)) \end{aligned} \tag{8}$$

where $\boldsymbol{y}_e(k)$ is the error between the real measurement output $\boldsymbol{y}(k)$ and the measurement output $\overline{\boldsymbol{y}}(k)$ for the nominal system and is defined as

$$\boldsymbol{y}_e(k) \stackrel{\triangle}{=} \boldsymbol{y}(k) - \overline{\boldsymbol{y}}(k) \tag{9}$$

Using the estimated error vector $\hat{\boldsymbol{e}}(k)$, we assume that the compensation input $\boldsymbol{v}(k)$ is described as

$$\boldsymbol{v}(k) \stackrel{\triangle}{=} -F_e\hat{\boldsymbol{e}}(k) \tag{10}$$

where the matrix F_e is the compensation gain matrix. Therefore the configuration of the proposed robust control system can be shown in figure 1.

Furthermore we introduce an estimation error vector $\boldsymbol{\eta}(k) \stackrel{\triangle}{=} \boldsymbol{e}(k) - \hat{\boldsymbol{e}}(k)$ and an augmented vector $\boldsymbol{e}_\eta(k) \stackrel{\triangle}{=} (\ \hat{\boldsymbol{e}}^T(k)\ \ \boldsymbol{\eta}^T(k)\ )^T$, then from eqs.(7)$\sim$(10), the following augmented system is obtained.

$$\boldsymbol{e}_\eta(k+1) = A_{e_\eta}(\theta)\boldsymbol{e}_\eta(k) + A_x(\theta)\overline{\boldsymbol{x}}(k) \tag{11}$$

$$\begin{aligned} &A_{e_\eta}(\theta) \\ &= \begin{pmatrix} A + G_eC_e(\theta) - BF_e & G_eC(\theta) \\ A_e(\theta) - G_eC_e(\theta) & A(\theta) - G_eC(\theta) \end{pmatrix} \end{aligned} \tag{12}$$

$$A_x(\theta) = \begin{pmatrix} 0 \\ A_e(\theta) \end{pmatrix} \tag{13}$$

where $C_e(\theta)$ is the matrix given by $C_e(\theta) = C(\theta) - C$. For the augmented system eq.(10), we define the following quadratic cost function

$$\begin{aligned} J_{e_\eta} = \sum_{k=0}^{\infty} \big(&\boldsymbol{e}^T(k)Q_e\boldsymbol{e}(k) + \boldsymbol{\eta}^T(k)Q_\eta\ \boldsymbol{\eta}(k) \\ &+\boldsymbol{v}^T(k)R_e\boldsymbol{v}(k)\big) \end{aligned} \tag{14}$$

the weighting matrices $Q_e \in \Re^{n\times n}$, $Q_\eta \in \Re^{n\times n}$ and $R_e \in \Re^{m\times m}$ are positive definite and can be adjusted so as to achieve the desirable response.

Our control objective in this paper is not only to ensure robust stability but also to achieve the satisfactory transient behavior in time response which is close to the desirable trajectory generated by the nominal system without using excessive control input. That is to design the compensation input and the observer by minimizing an upper bound on the quadratic cost function eq.(14). The conditions for the existence of the compensation input and the observer are derived in the next section.

3 Design of the Observer-Based Controller

In this section, we consider to design the compensation input and the observer which minimize the upper bound on the quadratic cost function eq.(14). In other words, we deal with the design problem of the compensation gain F_e and the observer gain G_e minimizing the upper bound on the quadratic cost function eq.(14). However, it is difficult to design the compensation gain F_e and the observer gain G_e simultaneously, because the uncertain parameters exist. Thus, firstly, we design the observer gain G_e. Next, we derive the condition for the existence of the compensation gain F_e to minimize the upper bound on the quadratic cost functioneq.(14).

3.1 Design of the Observer

From eqs.(7) and (8), the folllowing estimation error system is obtained.

$$\begin{aligned}\boldsymbol{\eta}(k+1) = &(A(\theta) - G_eC(\theta))\boldsymbol{\eta}(k) \\ &+(A_e(\theta) - G_eC_e(\theta))\hat{\boldsymbol{e}}(k) \\ &+(A_e(\theta) - G_eC_e(\theta))\overline{\boldsymbol{x}}(k) \quad (15)\end{aligned}$$

Now by ignoring the $\hat{\boldsymbol{e}}(k)$ and $\overline{\boldsymbol{x}}(k)$ in eq.(15), the following system is obtained.

$$\overline{\boldsymbol{\eta}}(k+1) = (A(\theta) - G_eC(\theta))\overline{\boldsymbol{\eta}}(k) \quad (16)$$

In this paper, we consider to design the observer gain G_e which quadratically stabilize the system eq.(16). We introduce the quadratic function $V(\overline{\boldsymbol{\eta}}, k) = \overline{\boldsymbol{\eta}}^T(k)Y_\eta\overline{\boldsymbol{\eta}}(k)$ as a Lyapunov function candidate and consider $\Delta V(\overline{\boldsymbol{\eta}}, k) \stackrel{\triangle}{=} V(\overline{\boldsymbol{\eta}}, k+1) - V(\overline{\boldsymbol{\eta}}, k)$ along the trajectory of the system eq.(16). From eq.(16), $\Delta V(\overline{\boldsymbol{\eta}}, k)$ can be computed as

$$\Delta V(\overline{\boldsymbol{\eta}}, k) = \overline{\boldsymbol{\eta}}^T(k)\Phi_\eta(\theta)\overline{\boldsymbol{\eta}}(k) \quad (17)$$

$$\Phi_\eta(\theta) = (A(\theta) - G_eC(\theta))^TY_\eta(A(\theta) - G_eC(\theta)) - Y_\eta \quad (18)$$

where Y_η is the symmetric positive definite matrix.

Therefore if the matrices G_e and Y_η satifying the condition $\Phi_\eta(\theta) < 0$ for $\forall\theta \in \Delta$, then the quadratic stability of the system eq.(16) is ensured. Considering the the set of vertices of the Δ such as $\Delta_{\text{vex}} \stackrel{\triangle}{=} \{\theta \in \Re^N \mid \theta_i = 1 \text{ for } i = 1, \cdots, N\}$, the design problem of the observer gain G_e is reduced to the problem of finding the matrices Y_η and V_η which satisfy the condition

$$\begin{pmatrix} -Y_\eta & A^T(\theta)Y - C^T(\theta)V_\eta^T \\ YA(\theta) - VC(\theta) & -Y_\eta \end{pmatrix} < 0 \quad \text{for } \forall\theta \in \Delta_{\text{vex}} \quad (19)$$

where V_η is a matrix satisfying $V_\eta = Y_\eta G_e$. Thus, if the solution of the linear matrix inequality (LMI) condition eq.(19) exists, then using the solution, the observer gain G_e can be obtained as

$$G_e = Y_\eta^{-1}V_\eta \quad (20)$$

3.2 Design of the Compensation Input

In the above design procedure, the observer gain G_e have been derived. Hence, we consider to design the compensation gain F_e minimizing the upper bound on the quadratic cost function eq.(14).

Now, using the vectors $\hat{\boldsymbol{e}}(k)$ and $\boldsymbol{\eta}(k)$ and the compensation input eq.(10), the quadratic cost function eq.(14) is rewritten as

$$J_{e_\eta} = \sum_{k=0}^{\infty} \boldsymbol{e}_\eta^T(k)\Theta_{e_\eta}\boldsymbol{e}_\eta(k) \quad (21)$$

where Θ_{e_η} is the matrix expressed as

$$\Theta_{e_\eta} = \begin{pmatrix} Q_e + F_e^TR_eF_e & Q_e \\ Q_e & Q_e + Q_\eta \end{pmatrix} \quad (22)$$

By introducing two positive definite matrices $\mathcal{X}_{e_\eta} \stackrel{\triangle}{=} diag(X_e, X_\eta)$ $(X_e, X_\eta \in \Re^{n\times n})$ and $\mathcal{P}_{e_\eta} \in \Re^{n\times n}$, the quadratic cost function eq.(22) can be expressed as

$$\begin{aligned}J_e = &\sum_{k=0}^{\infty} \begin{pmatrix} \boldsymbol{e}_\eta(k) \\ \overline{\boldsymbol{x}}(k) \end{pmatrix}^T \Phi_e(\theta) \begin{pmatrix} \boldsymbol{e}_\eta(k) \\ \overline{\boldsymbol{x}}(k) \end{pmatrix} \\ &-\boldsymbol{e}_\eta(\infty)\mathcal{X}_{e_\eta}\boldsymbol{e}_\eta(\infty) - \overline{\boldsymbol{x}}^T(\infty)\mathcal{P}_{e_\eta}\overline{\boldsymbol{x}}(\infty) \\ &+\boldsymbol{e}_\eta^T(0)\mathcal{X}_{e_\eta}\boldsymbol{e}_\eta(0) + \overline{\boldsymbol{x}}^T(0)\mathcal{P}_{e_\eta}\overline{\boldsymbol{x}}(0) \quad (23)\end{aligned}$$

In eq.(23), $\Phi_e(\theta)$ is a matrix expressed as

$$\Phi_e(\theta) = \begin{pmatrix} \Phi_{e_{11}}(\theta) & \Phi_{e_{12}}(\theta) \\ \Phi_{e_{12}}^T(\theta) & \Phi_{e_{22}}(\theta) \end{pmatrix} \quad (24)$$

$$\begin{aligned}\Phi_{e_{11}}(\theta) &= A_{e_\eta}^T(\theta)\mathcal{X}_{e_\eta}A_{e_\eta}(\theta) - \mathcal{X}_{e_\eta} + \Theta_{e_\eta} \\ \Phi_{e_{12}}(\theta) &= A_{e_\eta}^T(\theta)\mathcal{X}_{e_\eta}A_x(\theta) \\ \Phi_{e_{22}}(\theta) &= (A - BK)\mathcal{P}_{e_\eta}(A - BK) - \mathcal{P}_{e_\eta} \\ &\quad +A_x^T(\theta)\mathcal{X}_{e_\eta}A_x(\theta)\end{aligned} \quad (25)$$

Noting that, since the matrices Q_e, Q_η and R_e are positive definite, the quadratic cost function eq.(23) takes the positive value (i.e. $J_{e_\eta} \geq 0$) and the nominal system is asymptotically stable, i.e. $\overline{\boldsymbol{x}}(\infty) = \mathbf{0}$, if there exist $F_e, \mathcal{X}_{e\eta} > 0$ and $\mathcal{P}_{e_\eta} > 0$ such that

$$\Phi_e(\theta) \leq 0 \quad \text{for} \quad \forall\theta \in \Delta \quad (26)$$

then the following augmented system is quadratically stable because the quadratic function $\mathcal{V}(\boldsymbol{e}_\eta, \overline{\boldsymbol{x}}, k) \stackrel{\triangle}{=} \boldsymbol{e}_\eta^T(k)\mathcal{X}_{e_\eta}\boldsymbol{e}_\eta(k) + \overline{\boldsymbol{x}}^T(k)\mathcal{P}_{e_\eta}\overline{\boldsymbol{x}}(k)$ become a Lyapunov function. This fact is obtained by a few manipulations and using eqs.(27) and (28).

$$\begin{pmatrix} \boldsymbol{e}_\eta(k+1) \\ \overline{\boldsymbol{x}}(k+1) \end{pmatrix} = \mathcal{A}_{e_\eta}(\theta)\begin{pmatrix} \boldsymbol{e}_\eta(k) \\ \overline{\boldsymbol{x}}(k) \end{pmatrix} \quad (27)$$

$$\mathcal{A}_{e_\eta}(\theta) = \begin{pmatrix} A_{e_\eta}(\theta) & A_x(\theta) \\ 0 & A - BK \end{pmatrix} \quad (28)$$

$$\Psi_e(\theta) = \left(\begin{array}{c|c} -\mathcal{S}_{e_\eta} & 0 \\ \hline 0 & A_K\mathcal{P}_{e_\eta}A_K - \mathcal{P}_{e_\eta} \end{array}\right) + \left(\begin{array}{c|c} A_{e_\eta}(\theta)\mathcal{S}_{e_\eta} & A_x(\theta) \\ \hline \begin{matrix} S_e & 0 \\ 0 & S_\eta \\ W_e & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right)^T \left(\begin{array}{c|c} \mathcal{S}_{e_\eta}^{-1} & 0 \\ \hline 0 & \Theta^*_{e_\eta} \end{array}\right) \left(\begin{array}{c|c} A_{e_\eta}(\theta)\mathcal{S}_{e_\eta} & A_x(\theta) \\ \hline \begin{matrix} S_e & 0 \\ 0 & S_\eta \\ W_e & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right) \leq 0 \quad \text{for} \quad \forall\theta\in\Delta \tag{33}$$

$$\Psi^*_e(\theta) \triangleq \left(\begin{array}{c|c|c|c} -\mathcal{S}_{e_\eta} & 0 & \mathcal{S}_{e_\eta}A^T_{e_\eta}(\theta) & \begin{matrix} S_e & 0 & W_e^T \\ 0 & S_\eta & 0 \end{matrix} \\ \hline 0 & \mathcal{P}_{A_K} & A_x^T(\theta) & 0 \\ \hline A_{e_\eta}(\theta)\mathcal{S}_{e_\eta} & A_x(\theta) & -\mathcal{S}_{e_\eta} & 0 \\ \hline \begin{matrix} S_e & 0 \\ 0 & S_\eta \\ W_e & 0 \end{matrix} & 0 & 0 & -(\Theta^*_{e_\eta})^{-1} \end{array}\right) \leq 0 \quad \text{for} \quad \forall\theta\in\Delta \tag{35}$$

Therefore from the relations $\boldsymbol{e}_\eta(\infty) = \boldsymbol{0}$ and $\boldsymbol{e}(k) \triangleq \boldsymbol{x}(k) - \overline{\boldsymbol{x}}(k)$, the asymptotic stability of the uncertain system eq.(1) is guaranteed. Furthermore the quadratic cost function eq.(23) is bounded and the upper bound on the quadratic cost function eq.(23), denoted by $\mathcal{J}_{e_\eta}$, is written as

$$\begin{aligned} J_{e_\eta} &\leq \boldsymbol{e}^T_\eta(0)\mathcal{X}_{e_\eta}\boldsymbol{e}_\eta(0) + \overline{\boldsymbol{x}}^T(0)\mathcal{P}_{e_\eta}\overline{\boldsymbol{x}}(0) \\ &\triangleq \mathcal{J}_{e_\eta} \quad \text{for} \quad \forall\theta\in\Delta \end{aligned} \tag{29}$$

Now we consider to design the compensation gain which minimizes the upper bound on the cost function eq.(29). In eq.(25), however, the upper bound $\mathcal{J}_{e_\eta}$ depends on the initial values $\boldsymbol{e}_\eta(0)$ and $\overline{\boldsymbol{x}}(0)$. However since the state $\boldsymbol{x}(k)$ cannot be measured completely, the compensation gain matrix F_e cannot be designed by using the vector $\boldsymbol{e}_\eta(0)$. In order to avoid these dependences, we introduce the matrix $\mathcal{X}^*_{e_\eta} \triangleq diag(\mathcal{X}_{e_\eta}, \mathcal{P}_{e_\eta})$ and assume that the initial values $\boldsymbol{e}_\eta(0)$ and $\overline{\boldsymbol{x}}(0)$ are zero mean random vector satisfying $E\{\boldsymbol{e}_\eta(0)\boldsymbol{e}^T_\eta(0)\} = I_{2n}$, $E\{\boldsymbol{e}_\eta(0)\} = \boldsymbol{0}$, $E\{\overline{\boldsymbol{x}}(0)\overline{\boldsymbol{x}}^T(0)\} = I_n$ and $E\{\overline{\boldsymbol{x}}(0)\} = \boldsymbol{0}$, respectively. In this case, we consider the value of the quadratic cost function as its expectation. Then the upper bound on the quadratic cost function eq.(29) is given as $E\{\mathcal{J}_{e_\eta}\} = Tr\{\mathcal{X}^*_{e_\eta}\}$. Therefore we seek to minimize $Tr\{\mathcal{X}^*_{e_\eta}\}$ subject to the constraint eq.(26). Namely the problem of designing the compensation input to minimize the upper bound on the cost function eq.(29) is reduced to the following constrained optimization problem

$$\begin{aligned} &\underset{\mathcal{X}_{e_\eta}, F_e, \mathcal{P}_{e_\eta}}{\text{Min}} [Tr\{\mathcal{X}^*_{e_\eta}\}] \quad \text{subject to} \\ &\qquad \text{eq.(23)}, \ \mathcal{X}_{e_\eta} > 0 \ \text{ and } \ \mathcal{P}_{e_\eta} > 0 \end{aligned} \tag{30}$$

We introduce the matrix $S_{e_\eta} \triangleq diag(S_e, S_\eta) = X^{-1}_{e\eta}$ and consider the change of variable $W_e \triangleq F_e S_e$. Then pre- and post-multiplying eq.(26) by $diag(\mathcal{S}_{e_\eta}, I_n)$, the condition eq.(26) is written as

$$\begin{aligned} \Psi_e(\theta) &= \begin{pmatrix} \Psi_{e_{11}}(\theta) & \Psi_{e_{12}}(\theta) \\ \Psi^T_{e_{12}}(\theta) & \Psi_{e_{22}}(\theta) \end{pmatrix} \\ &\leq 0 \quad \text{for} \quad \forall\theta\in\Delta \end{aligned} \tag{31}$$

where $\Psi_{e_{ij}}(\theta)$ $(i,j=1,2)$ are the matrices given by

$$\begin{aligned} \Psi_{e_{11}}(\theta) &= \mathcal{S}_{e_\eta}A^T_{e_\eta}(\theta)\mathcal{S}^{-1}_{e_\eta}A_{e_\eta}(\theta)\mathcal{S}_{e_\eta} - \mathcal{S}_{e_\eta} \\ &\quad + \mathcal{S}_{e_\eta}\Theta_{e_\eta}\mathcal{S}_{e_\eta} \\ \Psi_{e_{12}}(\theta) &= \mathcal{S}_{e_\eta}A^T_{e_\eta}(\theta)\mathcal{X}_{e_\eta}A_x(\theta) \\ \Psi_{e_{22}}(\theta) &= (A-BK)\mathcal{P}_{e_\eta}(A-BK) - \mathcal{P}_{e_\eta} \\ &\quad + A^T_x(\theta)\mathcal{S}^{-1}_{e_\eta}A_x(\theta) \end{aligned} \tag{32}$$

Furthermore, the inequality eq.(31) can be described as eq.(33). In eq.(33) , A_K is the matrix given by $A_K = A - BK$ and the matrix $\Theta^*_{e_\eta}$ is expressed by

$$\Theta^*_{e_\eta} = \begin{pmatrix} Q_e & Q_e & 0 \\ Q_e & Q_e + Q_\eta & 0 \\ 0 & 0 & R_e \end{pmatrix} \tag{34}$$

Note that the matrix $\Theta^*_{e_\eta}$ is positive definite, because using the Schur complement formula, positive definiteness of $Q_{e\eta}$ is equivalent to $Q_e - Q_e(Q_e + Q_\eta)^{-1}Q_e > 0$.

Since the matrix $\Theta^*_{e_\eta}$ is positive definite, we can apply the Schur complement formula to the condition eq.(33). By using the Schur complement, the condition eq.(33) is equivalent to the inequality eq.(35). and the inequality eq.(35) is LMI in S_e, S_η, W_e and $\mathcal{P}_{e_\eta}$ because the matrix $A_{e_\eta}(\theta)\mathcal{S}_{e_\eta}$ is expressed as

$$A_{e_\eta}(\theta)\mathcal{S}_{e_\eta} = \begin{pmatrix} A_{e_{11}}(\theta) & A_{e_{12}}(\theta) \\ A_{e_{21}}(\theta) & A_{e_{22}}(\theta) \end{pmatrix} \tag{36}$$

$$\begin{aligned} A_{e_{11}}(\theta) &= AS_e + G_eC_e(\theta)S_e - BW_e \\ A_{e_{12}}(\theta) &= G_eC(\theta)S_\eta \\ A_{e_{21}}(\theta) &= (A_e(\theta) - G_eC_e(\theta))S_e \\ A_{e_{22}}(\theta) &= (A(\theta) - G_eC(\theta))S_\eta \end{aligned} \tag{37}$$

Thus the condition eq.(35) can be transformed into

$$\Psi^*_e(\theta) \leq 0 \quad \text{for} \quad \forall\theta\in\Delta_{\text{vex}} \tag{38}$$

Here introducing two complementary variables $\mathcal{Z}_{e_\eta} \in \Re^{2n\times 2n}$ and $\mathcal{Z}^*_{e_\eta} \in \Re^{3n\times 3n}$

$$\begin{pmatrix} \mathcal{Z}_{e_\eta} & I_{2n} \\ I_{2n} & \mathcal{S}_{e_\eta} \end{pmatrix} \geq 0 \tag{39}$$

$$\mathcal{Z}^*_{e_\eta} \triangleq diag(\mathcal{Z}_{e_\eta}, \mathcal{P}_{e_\eta}) \tag{40}$$

the minimization problem of $Tr\{\mathcal{X}^*_{e_\eta}\}$ can be transformed into that of $Tr\{\mathcal{Z}^*_{e_\eta}\}$. The condition eq.(39) is also the linear matrix inequality (LMI) in $\mathcal{Z}_{e_\eta}$ and $\mathcal{S}_{e_\eta}$. Consequently, the constrained optimization problem eq.(30) is reduced to the following constrained convex optimization problem.

$$\min_{\mathcal{Z}_{e_\eta}, F_e, W_e, \mathcal{S}_{e_\eta}, \mathcal{P}_{e_\eta}} [Tr\{\mathcal{Z}^*_{e_\eta}\}] \text{ subject to}$$
$$\text{eqs.(38) and (39)}, \quad \mathcal{S}_{e_\eta} > 0 \text{ and } \mathcal{P}_{e_\eta} > 0 \tag{41}$$

If the optimal solution of the optimization problem eq.(41) is obtained, then using the optimal solution, the compensation gain F_e can be computed as

$$F_e = W_e S_e^{-1} \tag{42}$$

Consequently, the following theorem for designing the compensation input and the observer is obtained.

Theorem 1 *There exists the compensation input minimizing the upper bound on the quadratic cost function eq.(29), if there exist* $\mathcal{Z}_{e_\eta} > 0, W_e, \mathcal{S}_{e_\eta} > 0$ *and* $\mathcal{P}_{e_\eta} > 0$ *such that*

$$\min_{\mathcal{Z}_{e_\eta}, F_e, W_e, \mathcal{S}_{e_\eta}, \mathcal{P}_{e_\eta}} [Tr\{\mathcal{Z}^*_{e_\eta}\}] \text{ subject to}$$
$$\text{eqs.(38) and (39)}, \quad \mathcal{S}_{e_\eta} > 0 \text{ and } \mathcal{P}_{e_\eta} > 0$$

where $\mathcal{Z}^*_{e_\eta}, \mathcal{S}_{e_\eta}$ *and* W_e *are the matrices expressed as* $\mathcal{Z}^*_{e_\eta} \triangleq diag(\mathcal{Z}_{e_\eta}, P_{e_\eta})$, $\mathcal{S}_{e_\eta} \triangleq diag(S_e, S_\eta), W_e \triangleq F_e S_e$ *and the matrices* $A_{e_\eta}(\theta)\mathcal{S}_{e_\eta}$ *and* $A_x(\theta)$ *are given by eqs.(13),(36) and (37) and the observer gain* G_e *is derived as*

$$G_e = Y_\eta^{-1} V_\eta$$

using the solution of the linear matrix inequality (LMI) condition eq.(19).

If the optimal solution $\mathcal{Z}_{e_\eta} > 0, W_e, \mathcal{S}_{e_\eta} > 0$ *and* $\mathcal{P}_{e_\eta} > 0$ *of the optimization problem is obtained, using the solution of the constrained convex optimization problem, the compensation input* $\boldsymbol{v}(k)$

$$\boldsymbol{v}(k) \triangleq -F_e \hat{\boldsymbol{e}}(k)$$
$$F_e = W_e S_e^{-1}$$

4 Numerical Example

In this section, we illustrate the effectiveness of the proposed robust controller by the following simple example. The simulation result is shown for the proposed control law and an observer-based guaranteed cost control law based on the work of Petersen and McFarlane 1994[1].

Consider the following linear discrete-time system with an unknown parameter.

$$\boldsymbol{x}(k+1) = \begin{pmatrix} 1+\delta^* & 2 \\ 0 & 0.8 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(k)$$
$$\boldsymbol{y}(k) = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}(k)$$

where the parameter δ^* is the uncertainties and is assumed to vary within the interval $[-0.25, 0.25]$.

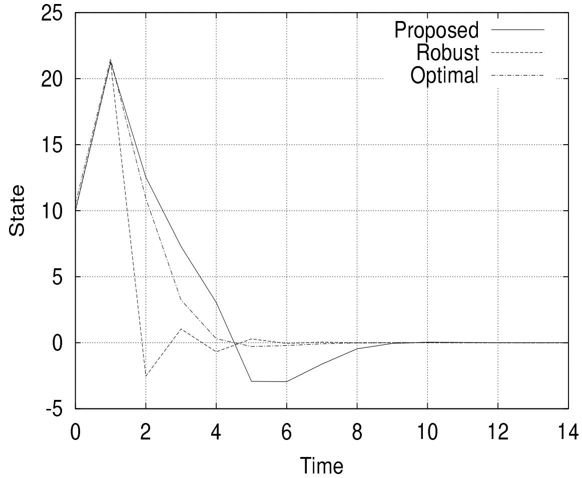

Fig. 2: Time histories of the state $x_1(t)$

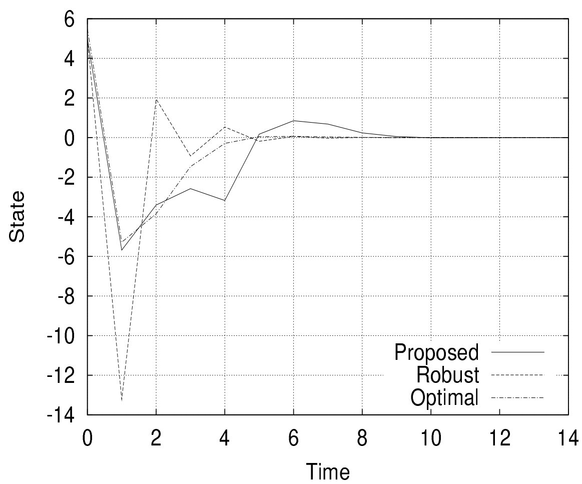

Fig. 3: Time histories of the state $x_2(t)$

Now for the nominal system, the weighting matrices in the quadratic cost function for the linear quadratic control problem are selected in advance as $Q = 1.0 I_2$ and $R = 9.0$ in order to achieve the desirable transient behavior. Solving the algebraic Riccati equation eq.(5), we obtain the following optimal feedback gain for the nominal system

$$K = \begin{pmatrix} 0.1413 & 0.6107 \end{pmatrix}$$

Furthermore, by solving the linear matrix inequality (LMI) condition eq.(refeq19), the observer gain G_e is derived as

$$G_e = \begin{pmatrix} 1.651 \\ 0.2585 \end{pmatrix}$$

Note that, for the guaranteed cost control, the state vector is estimated by using the observer

$$\hat{\boldsymbol{x}}(k+1) = (A - BK_r)\hat{\boldsymbol{x}}(k) + G_e(\boldsymbol{y}(k) - C\hat{\boldsymbol{x}}(k))$$

i.e. the observer for the guaranteed cost control has same structure as those for the proposed control.

To design the compensation gain F_e minimizing the upper bound on the quadratic cost function

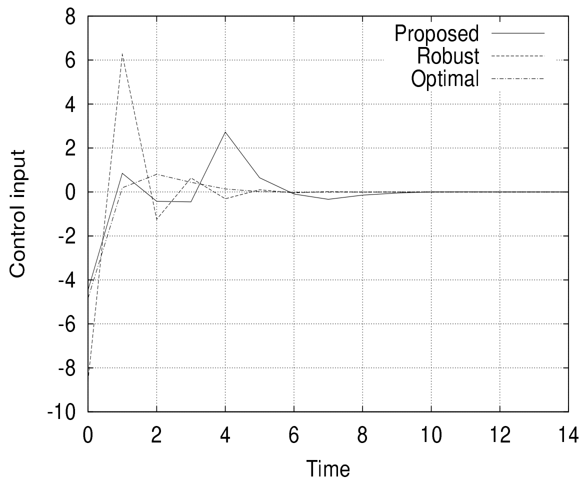

Fig. 4: Time histories of the control input $u(t)$

eq.(29), we solve the constrained convex optimization problem eq.(41). We choose the weighting matrices $Q_e = I_2, Q_\eta = 4.0I_2$ and $R_e = 4.0$, then the compensation gain is given as

$$F_e = (\ 0.30471 \ \ 0.99767\ )$$

i.e. the proposed control input is described as

$$\begin{aligned} u(k) &= -(\ 0.1413 \ \ 0.6107\ )\overline{\boldsymbol{x}}(k) \\ &\quad -(\ 0.30471 \ \ 0.99767\ )\hat{\boldsymbol{e}}(k) \end{aligned}$$

On the other hand, the feedback gain for the guaranteed cost control have been obtained as

$$K_r = (\ 0.2999 \ \ 0.9927\ )$$

We assume that the initial states for the uncertain system and the nominal system are selected as $\boldsymbol{x}(0) = (\ 10.0 \ \ 5.0\ )^T$ and $\overline{\boldsymbol{x}}(0) = (\ 10.5 \ \ 5.5\ )^T$, respectively, i.e. the initial error vector $\boldsymbol{e}(0) = (\ -0.5 \quad -0.5\ )^T$ because the complete state of the uncertain system cannot be measured. Furthermore, the initial values for the observers are chosen as $\hat{\boldsymbol{e}}(0) = (\ 0.0 \ \ 0.0\ )^T$ and $\hat{\boldsymbol{x}}(0) = (\ 10.5 \ \ 5.5\ )^T$, respectively.

In this example, we consider the following case that the real region of the uncertainties is smaller than the estimated perturbation region.

- $\delta^* = 0.125$

The simulation result is shown in figures 2-4. In these figures, Optimal and Robust represent the desirable transient behavior (i.e. the LQ optimal control trajectory generated by using the nominal system in this paper) and the time response for the observer-based guaranteed cost control law respectively.

From figures 2 and 3, we find that the proposed robust control law reflects the effect on the uncertainties and show good control performance close to the given transient response compared with the guaranteed cost control law (i.e. Robust in figures). Furthermore, from figure 4, the magnitude of the proposed control input is satisfactory comparing with the optimal control input. On the other hand, the magnitude of the observer-based guaranteed cost control law is excessive. Thus the effectiveness of the proposed robust controller is shown. This result shows that the proposed control law reflects the real effect due to the unknown parameter δ^*.

5 Conclusions

In this paper, for linear discrete-time systems with structured uncertainties, we present an observer-based robust controller design method which suppresses the deterioration of control performance when we compare the transient behavior in time response for an uncertain system with a desirable transient response generated by using the nominal system without using excessive control input.

Although the proposed controller include the nominal dynamics, the proposed controller design method is a simple approach due to the application of the linear quadratic control problem and reduces the cautiousness in a robust controller with a constant gain. In other words, though the structure of the proposed controller is complex comparing with a single controller with a fixed gain, the proposed design method can construct more flexible and adaptive robust control system, because using the error information is equivalent to giving consideration to the effect on the parameter uncertainties as on-line information. Furthermore, by using LMI solver such as LMI Control Toolbox in MATLAB and LMITOOL in Scilab, the proposed observer-based controller can be obtained easily.

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