Descriptor and Non-Descriptor Controllers in Mixed H_2/H_∞ Control of Descriptor Systems

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Abstract: This paper considers the design of mixed H_2/ H_∞ controllers for linear time-invariant descriptor systems. Firstly, an H_∞ and H_2 synthesis problem for a descriptor system are presented separately in terms of linear matrix inequalities (LMIs) based on the bounded real lemma. Then, the existence of a mixed H_2/ H_∞ controller by which the H_2 norm of the second channel is minimized while keeping the H_∞ norm bound of the first channel less than γ, is reduced to the linear objective minimization problem. The class of desired controllers that are assumed to have the same structure as the plant is parameterized by using the linearizing change of variables. In addition, we show the procedure by which a obtained descriptor controller can be transformed to a non-descriptor one.

Keywords: Descriptor system, Mixed H_2/ H_∞ control, LMI, Non-descriptor controller

1. INTRODUCTION

The descriptor system model has a high ability in representing dynamical systems. It can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even the improper part of the system in the same form. Models of chemical processes, for example, typically consist of differential equations describing the dynamic balance of mass and energy (that describes static constraints on physical variables) while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. (these are static and impulsive parts of the physical system). In other words, the descriptor model is much superior to the state-space one.

To design a controller for the descriptor system by using the general methods for the non-descriptor, requires modification of a descriptor system to a general state-space representation. That modification, however, necessarily causes a loss of information in the original descriptor system. In recent years, due to this fact, much work has been focused on analysis and design techniques for descriptor systems (see [1]).

In this paper, we consider the mixed H_2/ H_∞ control problem for descriptor systems. That is, the goal of this problem is to minimize bounds on the H_2-norm of the second channel $(T_2 : w_2 \to z_2)$, while keeping the H_∞ norm bound of the first channel $(T_1 : w_1 \to z_1)$ less than γ , i.e.,

$$\min\|T_2\|_2 \text{ subject to } \|T_1\|_\infty < \gamma$$

The achieved controller, however, is to have the same structure as the plant, i.e. a descriptor system. In case of realizing the controller, non-descriptor one is preferred. So, under some conditions, we can have a non-descriptor controller by way of SVD transformation on the given plant.

The paper is organized as follows. Section 2 gives the necessary background and some results related to the stability of descriptor systems. In section 3, we propose an LMI formulation for the H_∞ and H_2 synthesis problem by means of an adaptation of the 'linearizing change of variables [8]. These results, in the sequel, show LMI algorithms of the mixed H_2/ H_∞ control problem for descriptor systems. In section 4, we confirm the validity of the proposed method through two numerical examples.

2. MIXED H_2/ H_∞ CONTROL PROBLEM FOR LINEAR DESCRIPTOR SYSTEMS

2.1 Linear Descriptor Systems

Let us consider a linear time-invariant descriptor system

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{w},\quad \boldsymbol{z} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{w} \tag{1}$$

where $\boldsymbol{x} \in R^n$ is the descriptor variable, $\boldsymbol{w} \in R^q$ is input, $\boldsymbol{z} \in R^p$ is output, and $\boldsymbol{E},\boldsymbol{A} \in R^{n\times n}$, $\boldsymbol{B} \in R^{n\times q}, \boldsymbol{C} \in R^{p\times n}$, $\boldsymbol{D} \in R^{p\times q}$ are constant matrices. The matrix $\boldsymbol{E}$ may be singular and we denote its rank by $r = \text{rank}\boldsymbol{E} \le n$. As a shorthand notation for system (1) we often write $(\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$(or $(\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ if $\boldsymbol{D}=0$).

The system (1) has a unique solution for any initial conditions and constraints input $\boldsymbol{w}(\bullet)$ if $\det(s\boldsymbol{E}-\boldsymbol{A}) \ne 0$. In this case, (1) is said to be regular. On the other hand, a non-regular system always admits multiple solutions for the unforced $(\boldsymbol{w}=0)$ homogenous initial value problem. For a regular system (1), the transfer matrix

$$G(s) := \boldsymbol{C}(s\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{2}$$

can be defined. The question of impulsive solutions of regular systems is usually studied in terms of the Weierstrass canonical form (WCF) of $(\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$.

Theorem 1 [9]: Let $(\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$ be regular. Then there exists an equivalent system $(\tilde{\boldsymbol{E}},\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}},\tilde{\boldsymbol{C}},\tilde{\boldsymbol{D}}) \sim (\boldsymbol{E},\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$ with

$$\tilde{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{N} \end{bmatrix},\quad \tilde{\boldsymbol{A}} = \begin{bmatrix} \Lambda & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-r} \end{bmatrix} \tag{3}$$

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_s \\ \boldsymbol{B}_f \end{bmatrix},\quad \tilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C}_s & \boldsymbol{C}_f \end{bmatrix},\quad \tilde{\boldsymbol{D}} = \boldsymbol{D} \tag{4}$$

where matrix $\Lambda \in R^{r\times r}$ is of Jordan canonical form, and $\boldsymbol{N} \in R^{(n-r)\times(n-r)}$ is nilpotent. □

If (1) is in WCF, i.e.

$$\begin{bmatrix} \dot{\boldsymbol{x}}_s \\ \boldsymbol{N}\dot{\boldsymbol{x}}_f \end{bmatrix} = \begin{bmatrix} \Lambda & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_s \\ \boldsymbol{x}_f \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_s \\ \boldsymbol{B}_f \end{bmatrix}\boldsymbol{w} \tag{5}$$

then the part x_s (slow mode) of the descriptor vector $x^T = [x_s^T \quad x_f^T]$ is governed by an ordinary differential equation while

$$x_f = -\sum_{i=0}^{\nu-1} \delta^{(i)}(t) N^{i+1} x_f(0-) - \sum_{i=0}^{\nu-1} N^i B_f w^{(i)} \tag{6}$$

represents impulsive (fast) mode of the system (1) (with $\delta(t)$ the Dirac delta and the superscript (i) means the *i*th derivative). We conclude that descriptor systems will have no impulsive solutions (for all $w(\bullet) \in L_2[0,\infty)$ and all the initial conditions) if and only if their index is one.

2.2 Mixed H$_2$/ H$_\infty$ control problem

We consider a generalized plant G that is a descriptor system

$$G : \begin{cases} E\dot{x} = Ax + B_1 w + B_2 u \\ z_\infty = C_1 x \\ z_2 = C_2 x \\ y = C_3 x \end{cases} \tag{7}$$

where $x \in R^n$ denotes the descriptor variables, $u \in R^{m_2}$ the control input, $w \in R^{m_1}$ the disturbance input, $z_\infty \in R^{p_1}$ the external output related to H$_\infty$ control, $z_2 \in R^{p_2}$ the external output related to H$_2$ control, and $y \in R^{p_3}$ the measured output. A, B_i and C_i are constant matrices of appropriate dimension and E is a possibly singular matrix having the same dimension as A. Notice that in the descriptor setup there is no loss of generality in not considering a direct fed-through from control/disturbance input to external/measured output since the corresponding terms can be captured by additional descriptor variables ([1], refer example 1 of section 4).

The focus of the mixed H$_2$/ H$_\infty$ control problem is as follows: We want to find a dynamic output feedback controller K_E (with $\varsigma \in R^{n_K}$) in descriptor form

$$K_E : \begin{cases} E_K \dot{\varsigma} = A_K \varsigma + B_K y \\ u = C_K \varsigma + D_K y \end{cases} \tag{8}$$

such that

1. The closed-loop is regular and stable index one system. A system with these properties is said to be admissible [1].
2. The H$_2$ norm of the closed-loop transfer matrix $T_{cl2} : w \to z_2$ is minimized while keeping the H$_\infty$ norm of the closed-loop transfer matrix $T_{cl\infty} : w \to z_\infty$ less than γ.

In this paper, we assume two conditions related to the controller, that is, $\operatorname{rank} E_k = r$ and $n_K = n$. Under this circumstance, it is always possible to make $E_K = E$ without lose of generality by using proper similarity transformation [5].

3. H$_\infty$ CONTROL PROBLEM FOR DESCRIPTOR SYSTEM

3.1 Problem Setup

The plant from (7)

$$G_\infty : \begin{cases} E\dot{x} = Ax + B_1 w + B_2 u \\ z_\infty = C_1 x \\ y = C_3 x \end{cases} \tag{9}$$

together with the controller (8) forms a closed-loop system as follows:

$$\begin{aligned} E_{cl}\dot{x}_c &= A_{cl} x_c + B_{cl} w \\ z_\infty &= C_{cl\infty} x_c, \; x_c^T = [x^T \quad \varsigma^T] \end{aligned} \tag{10}$$

where

$$\mathbf{E}_{cl} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix}, \mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_2\mathbf{D}_K\mathbf{C}_3 & \mathbf{B}_2\mathbf{C}_K \\ \mathbf{B}_K\mathbf{C}_3 & \mathbf{A}_K \end{bmatrix} \tag{11}$$

$$\mathbf{B}_{cl} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}, \mathbf{C}_{cl\infty} = [\mathbf{C}_1 \quad \mathbf{0}]$$

The following theorem is based on a LMI characterization of admissibility and H$_\infty$ norm bound of the closed-loop system.

Theorem 2 [1](Generalized Bounded Real Lemma) A system $(E_{cl}, A_{cl}, B_{cl}, C_{cl\infty})$ is admissible and

$$\|G_{cl\infty}\|_\infty < \gamma, \; G_{cl\infty} := C_{cl\infty}(sE_{cl} - A_{cl})^{-1} B_{cl} \tag{12}$$

if and only if there exists matrix $\tilde{P}$ which satisfies

$$E_{cl}^T \tilde{P} = \tilde{P}^T E_{cl} \geq 0 \tag{13}$$

$$\begin{bmatrix} \mathbf{A}_{cl}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^T\mathbf{A}_{cl} & \tilde{\mathbf{P}}^T\mathbf{B}_{cl} & \mathbf{C}_{cl\infty}^T \\ \mathbf{B}_{cl}^T\tilde{\mathbf{P}} & -\gamma\mathbf{I} & \mathbf{0} \\ \mathbf{C}_{cl\infty} & \mathbf{0} & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{14}$$

Proof. This is a well-known result of LMI theory for descriptor systems. □

Theorem 2 represents a convenient tool for checking a H$_\infty$ norm bound of a descriptor system since it only requires the computation of the solution of the LMIs (13), (14), i.e. the solution of a feasibility problem.

3.2 LMI Synthesis

In view of theorem 2, the existence of matrices A_K, B_K, C_K, D_K, and $\tilde{P}$ such that matrix inequalities (13), (14) hold true is sufficient for the H$_\infty$ control problem. However (14) is nonlinear in these matrix variables and therefore difficult to solve. The idea in the following is to provide a linearizing change of variables along with the lines of [10] in order to end up with linear matrix inequalities instead of (13) and (14).

To present a strict LMI condition, we introduce matrices $V, U \in R^{n\times(n-r)}$ which are of full column rank and composed of bases of ($\ker E$) and ($\ker E^T$), respectively.

A possible solution $\tilde{P}$ of (14) is necessarily nonsingular, so we partition $\tilde{P}$ and $\tilde{P}^{-1}$ as

$$\tilde{P} = \begin{bmatrix} (QE + URV^T) & N \\ N^T & * \end{bmatrix} \tag{15-a}$$

$$\tilde{P}^{-1} = \begin{bmatrix} (PE^T + VSU^T) & M \\ M^T & * \end{bmatrix} \tag{15-b}$$

where $P, Q \in R^{n\times n}$ are symmetrical matrices, $S, R \in R^{(n-r)\times(n-r)}$ are nonsingular matrices, and $U, V \in R^{n\times(n-r)}$ were already defined in theorem 2. And N, M are nonsingular matrices with proper dimension. For the simplicity of expression, we temporarily adopt matrices X, Y such as

$$X = (PE^T + VSU^T), Y = (QE + URV^T) \tag{16}$$

Then from $\tilde{P}\tilde{P}^{-1} = I$, we obtain

$$\tilde{P}\Pi_1 = \Pi_2 \tag{17-a}$$

with

$$\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \; \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{17-b}$$

Since Π_1 is nonsingular, a nonsingular congruence transformation of (13), (14) is possible. That is,

$$\Pi_1^T E_{cl}^T \tilde{P}\Pi_1 = \Pi_1^T \tilde{P}^T E_{cl}\Pi_1 \geq 0 \quad (18)$$

$$\Psi_{\Pi_1}^T \begin{bmatrix} A_{cl}^T\tilde{P} + \tilde{P}^T A_{cl} & \tilde{P}^T B_{cl} & C_{cl\infty}^T \\ B_{cl}^T\tilde{P} & -\gamma I & 0 \\ C_{cl\infty} & 0 & -\gamma I \end{bmatrix} \Psi_{\Pi_1} < 0 \quad (19)$$

$$\Psi_{\Pi_1} := \mathrm{diag}(\Pi_1 \quad I \quad I)$$

Such a congruence transformation has been suggested in [10] in order to reveal the affine structure of underlying matrix inequalities. In order to carry out (18) and (19), we first define the change of variables as follows:

$$\begin{cases} \hat{D} = D_K \\ \hat{C} = C_K M^T + D_K C_3 X \\ \hat{B} = NB_K + YB_2 D_K \\ \hat{A} = NA_K M^T + NB_K C_3 X + Y^T B_2 C_K M^T \\ \quad + Y^T (A + B_2 D_K C_3) X \end{cases} \quad (20)$$

Note that the new variables $\hat{A}, \hat{B}, \hat{C}$ have dimensions $n\times n, n\times m_2, p_3 \times n$, respectively. Then the direct calculation of inequalities (18) and (19) leads to (21) and (22).

$$\begin{bmatrix} EPE^T & E \\ E^T & E^T QE \end{bmatrix} \geq 0 \quad (21)$$

$$\begin{bmatrix} AX + X^T A^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + (A + B_2\hat{D}C_3) & * & * \\ \hat{A} + (A + B_2\hat{D}C_3)^T & A^T Y + Y^T A + \hat{B}C_3 + (\hat{B}C_3)^T & * & * \\ B_1^T & (Y^T B_1)^T & -\gamma I & * \\ C_1 X & C_1 & 0 & -\gamma I \end{bmatrix} < 0 \quad (22)$$

By introducing full column matrices $E_L, E_R \in R^{n\times(n-r)}$ that satisfy $E = E_L E_R^T$, we can further simplify the inequality (21). That is, (21) can be rewritten as

$$\begin{bmatrix} E_L & 0 \\ 0 & E_R \end{bmatrix} \begin{bmatrix} E_R^T PE_R & I_r \\ I_r & E_L^T QE_L \end{bmatrix} \begin{bmatrix} E_L^T & 0 \\ 0 & E_R^T \end{bmatrix} \geq 0 \quad (23)$$

We have $\mathrm{rank}\left(\tilde{E}_{cl}^T \tilde{P}\right) = 2r$, and $E_R^T PE_R > 0, E_L^T QE_L > 0$ can be assumed without loss of generality. Since the rank of a matrix is invariant under nonsingular congruence transformation, (21) can be written equivalently as

$$\begin{bmatrix} E_R^T PE_R & I_r \\ I_r & E_L^T QE_L \end{bmatrix} > 0 \quad (24)$$

i.e., as strict inequality. The inequalities (22), (24) are LMIs in the matrix variables $P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}$, and they constitute sufficient conditions for the existence of a controller by which the H_∞ norm condition is satisfied.

Next, in order to retrieve controller matrices A_K, B_K, C_K, D_K from the results of LMIs (22), (24) through (20), it is required to compute nonsingular matrices M, N. The following theorem is needed.

Theorem 3: From the previous development we can assume without loss of generality that S, R are nonsingular, and P, Q are symmetrical and satisfy $E_R^T PE_R > 0, E_L^T QE_L > 0$. Then $X = (PE^T + VSU^T), Y = (QE + URV^T)$ is nonsingular.

Proof: Since $[E_L \quad U], [E_R \quad V]$ are nonsingular, we multiply $[E_R \quad V]^T$ in the left side and $[E_L \quad U]$ in the right side of X (or Y). Then we can obtain

$$\det X = \det[E_R \quad V]^T X [E_L \quad U]$$
$$= \det \begin{bmatrix} (E_R^T PE_R) E_L^T E_L & 0 \\ * & V^T VSU^T U \end{bmatrix} \neq 0$$

□

Now we can determine the coefficient matrices of the controller (8).

Theorem 4: Consider a plant (9) and a controller (8). There exists a full order controller (that is, $n_K = n$ and $\mathrm{rank}\, E_K = r$) such that the closed-loop system (10) is admissible and H_∞ norm is bounded by γ if and only if LMIs (22), (24) admits solution $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$. A controller (8) solving the H_∞ norm problem then is given by matrices A_K, B_K, C_K, D_K retrieving from (20) with nonsingular matrices M, N such that

$$MN^T = I - XY \quad (25)$$

Proof *Necessity*: If there exists a solution $\tilde{P}$ to the inequalities (13), (14) for the closed-loop system (10), we always express it as $\tilde{P}\Pi_1 = \Pi_2$ with nonsingular matrices Π_1, Π_2 as in (17-b). Therefore a nonsingular congruence transformation (18), (19) is possible. By introducing the nonlinear change of variables (20), we can confirm that the inequalities (18), (19) become (24), (22) with the result of theorem 4.

Sufficiency: Assume that we have a solution $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ to (22), (24). In order to show the validity of the obtained controller (in other words, whether the H_∞ norm condition of the closed-loop system is satisfied), we need to establish nonsingular matrices $\Pi_1, \Pi_2, \tilde{P}$ as in (15)~(17). Looking at the left upper block of (17-a), M and N should be chosen as (25). From theorem 4, we infer that $I - XY$ is nonsingular. Hence we can always find square and nonsingular M and N satisfying (25). After defining Π_1, Π_2 as in (17-b), we observe that these matrices are nonsingular, and we set $\tilde{P} = \Pi_2\Pi_1^{-1}$ to obtain (17-a). With nonsingular matrices M, N, and the solutions $\{P, Q, S, R, \hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ to LMIs (22), (24), we can now determine controller matrices D_K, C_K, B_K, A_K in this order through (20). Consequently, using the relationship (20), it is possible to express the inequalities (21), (22) by means of the matrices $\{P, Q, S, R, A_K, B_K, C_K, D_K\}$, i.e. to reverse the linearizing change of variables. Recalling that Π_1 is square and nonsingular, the congruence transformation (18), (19) can be reversed, and by theorem 2 we can confirm that the H_∞ norm bound is satisfied with $\tilde{P} = \Pi_2\Pi_1^{-1}$ and controller matrices A_K, B_K, C_K, D_K. Hence the obtained controller indeed leads to $\|G_{cl\infty}\|_\infty < \gamma$. □

We can sum up controller computation as follows:

1) Compute the matrices U, V, E_L, E_R
2) Solve the LMIs (22), (24).
3) Compute the nonsingular matrices M, N, which satisfy (25), then computation of controller matrices D_K, C_K, B_K, A_K in this order.

3.3 H_2 control for descriptor systems

Here, we think about an LMI approach for H_2 control of linear time-invariant descriptor systems. Let us consider the descriptor and its closed-loop system given in section 2:

$$G_2:\begin{cases} E\dot{x}=Ax+B_1w+B_2u \\ z_2=C_2x \\ y=C_3x \end{cases} \quad (26)$$

$$\begin{aligned} E_{cl}\dot{x}_c &= A_{cl}x_c+B_{cl}w \\ z_2 &= C_{cl2}x_c,\ C_{cl2}=\begin{bmatrix} C_2 & 0 \end{bmatrix} \end{aligned} \quad (27)$$

The H_2 norm for a descriptor system (27) is defined as

$$\begin{aligned} \|G_{cl2}(s)\|_2^2 &:= \|C_{cl2}(sE_{cl}-A_{cl})B_{cl}\|_2^2 \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\text{trace}\left(G_{cl2}(-j\omega)^T G_{cl2}(j\omega)\right)d\omega\right] \end{aligned} \quad (28)$$

which is finite if and only if

$$\lim_{s\to\infty} G_{cl2}(s)=0 \quad (29)$$

To ensure finiteness of the H_2 norm, we assume that the system (26) satisfies the following condition [12].

$$\ker C_2 \supseteq \ker E \quad (30)$$

Therefore the H_2 control problem is to obtain a controller by which the H_2 norm of the closed loop system is less than, say, ν. It is known that the H_2 norm of a descriptor system (28) can be computed as follows [9]:

$$\|G_{cl2}(s)\|_2^2=\text{trace}\left(C_{cl2}E_{cl}L_cC_{cl2}^T\right)=\text{trace}\left(C_{cl2}L_c^TE_{cl}^TC_{cl2}^T\right)<\nu \quad (31)$$

where L_c solves

$$A_{cl}L_c+L_c^TA_{cl}^T+B_{cl}B_{cl}^T=0 \quad (32\text{-a})$$

$$E_{cl}L_c=L_c^TE_{cl}^T \quad (32\text{-b})$$

Since $L_c<L$ for any L satisfying

$$A_{cl}L_c+L^TA_{cl}^T+B_{cl}B_{cl}^T<0 \quad (33)$$

It is readily verified that $\|G_{cl2}\|_2^2<\nu$ if and only if there exists $L>0$ satisfying (33) and $\text{trace}\left(C_{cl2}L^TE_{cl}^TC_{cl2}^T\right)<\nu$. With an auxiliary parameter W, we obtain the following result. That is, $G_{cl2}(s)$ is stable and $\|G_{cl2}\|_2^2<\nu$ if and only if there exists $\tilde{P}=L^{-1}$ and symmetric W such that

$$\begin{bmatrix} A_{cl}^T\tilde{P}+\tilde{P}^TA_{cl} & \tilde{P}^TB_{cl} \\ B_{cl}^T\tilde{P} & -I \end{bmatrix}<0 \quad (34\text{-a})$$

$$\begin{bmatrix} \tilde{P}^TE_{cl} & C_{cl2}^T \\ C_{cl2} & W \end{bmatrix}>0 \quad (34\text{-b})$$

$$\text{trace}(W)<\nu \quad (34\text{-c})$$

Now, we can derive a theorem, which is parallel to theorem 5 and related to the H_2 control problem.

Theorem 5 Consider a descriptor system (26) with (30) and a controller (8). There exits a full order controller (that is, $n_K=n$ and $\text{rank}\,E_K=r$) such that the closed-loop system is admissible and the H_2 norm is bounded by ν if and only if LMIs (35) admits solution $\{P,Q,S,R,\hat{A},\hat{B},\hat{C},\hat{D}\}$.

$$\begin{bmatrix} AX+X^TA^T+B_2\hat{C}+\left(B_2\hat{C}\right)^T & * & * \\ \hat{A}+\left(A+B_2\hat{D}C_3\right)^T & A^TY+Y^TA+\hat{B}C_3+\left(\hat{B}C_3\right)^T & * \\ B_1^T & \left(Y^TB_1\right)^T & -I \end{bmatrix}<0 \quad (35\text{-a})$$

$$\begin{bmatrix} E_R^TPE_R & I_r & E_R^TPC_2^T \\ I_r & E_L^TQE_L & \left(E_R^TE_R\right)^{-1}E_R^TC_2^T \\ C_2PE_R & C_2E_R\left(E_R^TE_R\right)^{-1} & W \end{bmatrix}>0 \quad (35\text{-b})$$

$$\text{trace}(W)<\nu \quad (35\text{-c})$$

where X,Y are given in (16).

Proof By simply applying the congruence transformation with $\text{diag}(\Pi_1\quad I)$ to (34-a), (35-a) can be obtained. To show (35-b), some matrix manipulations are needed. We, first, carry out the congruence transformation to (34-b). That is,

$$\begin{aligned} &\begin{bmatrix} \Pi_1^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \tilde{P}^TE_{cl} & C_{cl2}^T \\ C_{cl2} & W \end{bmatrix}\begin{bmatrix} \Pi_1 & 0 \\ 0 & I \end{bmatrix} \\ &=\begin{bmatrix} EPE^T & E & (C_2X)^T \\ E^T & E^TQE & C_2^T \\ C_2X & C_2 & W \end{bmatrix}>0 \end{aligned} \quad (36)$$

where we used (18) and the following relationships that were obtained through (13).

$$\begin{aligned} &EX=EPE^T,\ Y^TE=E^TQE \\ &Y^TEX+NEM^T=E^T \end{aligned} \quad (37)$$

From Schur Complement, (36) is equivalent to

$$\begin{bmatrix} EPE^T & E \\ E^T & E^TQE \end{bmatrix}>\begin{bmatrix} (C_2X)^T \\ C_2^T \end{bmatrix}W^{-1}\begin{bmatrix} C_2X & C_2 \end{bmatrix} \quad (38)$$

The left side of (38) can be rewritten as follows:

$$\begin{bmatrix} E_L & U & 0 & 0 \\ 0 & 0 & E_R & V \end{bmatrix}\begin{bmatrix} E_R^TPE_R & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 \\ I_r & 0 & E_L^TQE_L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} E_L^T & 0 \\ U^T & 0 \\ 0 & E_R^T \\ 0 & V^T \end{bmatrix} \quad (39)$$

Since $\begin{bmatrix} E_L & U \end{bmatrix},\begin{bmatrix} E_R & V \end{bmatrix}$ are nonsingular, we can check the following relationships:

$$\begin{bmatrix} E_L & U \end{bmatrix}\begin{bmatrix} \left(E_L^TE_L\right)^{-1}E_L^T \\ \left(U^TU\right)^{-1}U^T \end{bmatrix}=I \quad (40\text{-a})$$

$$\begin{bmatrix} E_R & V \end{bmatrix}\begin{bmatrix} \left(E_R^TE_R\right)^{-1}E_R^T \\ \left(V^TV\right)^{-1}V^T \end{bmatrix}=I \quad (40\text{-b})$$

Multiplying (39) by $\begin{bmatrix} E_L & U & 0 & 0 \\ 0 & 0 & E_R & V \end{bmatrix}^{-1}$ from the left and by its transpose from the right leads to

$$\begin{bmatrix} E_R^TPE_R & I_r \\ I_r & E_L^TQE_L \end{bmatrix}>\begin{bmatrix} E_R^TPC_2^T \\ \left(E_R^TE_R\right)^{-1}E_R^TC_2^T \end{bmatrix}W^{-1}\begin{bmatrix} C_2PE_R & C_2E_R\left(E_R^TE_R\right)^{-1} \end{bmatrix} \quad (41)$$

where we used (40). Again applying Schur complement to (41), (35-b) is obtained. We have shown that (34) are equivalent to (35). So the rest of the proof (that is, the existence of a controller) can be done in the same way as in theorem 5. □

3.4 LMI Conditions for Mixed H_2/ H_∞ control problem

The mixed H_2/ H_∞ control problem is to design a controller (8) such that: (1) internally stabilizing the closed-loop system and (2) minimize H_2 norm of the closed-loop while satisfying the given H_∞ norm bound. Therefore, by combining the results of the previous subsections, we can easily obtain the following theorem:

Theorem 6 Consider a descriptor system (7) with (30). There exists a full order controller (8) such that the closed loop system is admissible and the H_∞ norm is bounded by γ and the H_2 norm is less than ν if and only if LMIs (22), (24), and (35) admit solutions $\{P,Q,S,R,\hat{A},\hat{B},\hat{C},\hat{D}\}$. Then one of the controller (8) solving the mixed H_2/H_∞ control problem is given by matrices A_K,B_K,C_K,D_K through (20) with nonsingular matrices M, N satisfying (25). □

As a result of this theorem, the mixed H_2/ H_∞ control problem can be solved by minimizing ν under the LMI constraints. That is,

Minimize $\mathrm{trace}(W)$ subject to LMIs (22), (24), (34-a), and (34-b)

Since this is so called the linear objective minimization problem, it is easily solved by using commercial software such as [9].

4.NON-DESCRIPTOR CONTROLLERS

In the previous section, we have obtained an explicit representation of the controller matrices by using the solution of LMIs. The controller, however, is set up in descriptor form (8). Since the objectives of the mixed H_2/ H_∞ control problem are closed loop feature, the design procedure itself does not guarantee any further structural properties of the controller (the closed loop is guaranteed to be index one – the controller is not). Furthermore, in order to implement the controller as a non-descriptor system, the obtained one has to be inverted. Especially in cases where the original controller is high index, this will cause numerical problems. Here, we suggest a procedure for obtaining a non-descriptor controller.

Without loss of generality, we consider a plant (7) and a controller (8) in normalized SVD representation, that is, we assume

$$E = E_K = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, A_K = \begin{bmatrix} A_{K11} & A_{K12} \\ A_{K21} & A_{K22} \end{bmatrix} \tag{42}$$

and other matrices of the controller (8) are already decomposed according to the structure of (42). By applying the SVD representation to (15), we can fix structures of matrices N, M as follows:

$$N = \begin{bmatrix} N_1 & N_2 \\ 0 & N_3 \end{bmatrix}, M = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} \tag{43}$$

Related to the SVD representation, we need the following lemma[1].

[**Lemma**] Let (E_K, A_K) be the SVD form of (42). Then the pair (E_K, A_K) is impulse-free if and only if A_{K22} is nonsingular. □

We consider the controller (8), which was transformed to the SVD form:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \end{bmatrix} = \begin{bmatrix} A_{K11} & A_{K12} \\ A_{K21} & A_{K22} \end{bmatrix}\begin{bmatrix} x_{K1} \\ x_{K2} \end{bmatrix} + \begin{bmatrix} B_{K1} \\ B_{K2} \end{bmatrix} y \tag{44-a}$$

$$u = [C_{K1} \quad C_{K2}][x_{K1} \quad x_{K2}]^T + D_K y \tag{44-b}$$

If A_{K22} is nonsingular, a non-descriptor controller can be easily obtained as follows:

$$\begin{cases} \dot{x}_{K1} = \tilde{A}_K x_{K1} + \tilde{B}_K y \\ u = \tilde{C}_K x_{K1} + \tilde{D}_K y \end{cases} \tag{45-a}$$

$$\begin{aligned} &\tilde{A}_K := A_{K11} - A_{K12}A_{K22}^{-1}A_{K21}, \tilde{B}_K := B_{K1} - A_{K12}A_{K22}^{-1}B_{K2} \\ &\tilde{C}_K := C_{K1} - C_{K2}A_{K22}^{-1}A_{K21}, \tilde{D}_K := D_K - C_{K2}A_{K22}^{-1}B_{K2} \end{aligned} \tag{45-b}$$

Here, if we suppose the condition $p_3 \le n - r$, the feedthrough matrix D_K also can be realized by $\tilde{C}_{K2}, \tilde{D}_{K2}$, with $\tilde{C}_{K2}\tilde{D}_{K2} := D_K - C_{K2}A_{K22}^{-1}B_{K2}$ [12], i.e. without loss of generality we set $D_K = 0$. By using this assumption, (20) becomes

$$\begin{cases} \hat{C} = C_K M^T \\ \hat{B} = NB_K \\ \hat{A} = NA_K M^T + \hat{B}_K C_3 X + Y^T B_2 \hat{C}_K + Y^T AX \end{cases} \tag{46}$$

Due to theorem 5, the controllers with a structure of (44) are parameterized by solutions to LMIs (22) and (24). Since any index one controller is equivalent to (44), the remaining thing is to show that, any specific solutions to the LMIs (22), (24), it is possible to choose M, N so that the controller takes the form of (44).

First, in order guarantee the non-singularity of A_{K22}, we select $A_{K22} = I_{n-r}$. Therefore, matrices M, N are determined to satisfy the following equations:

$$\begin{bmatrix} N_1 & N_2 \\ 0 & N_3 \end{bmatrix}\begin{bmatrix} A_{K11} & A_{K12} \\ A_{K21} & I_{n-r} \end{bmatrix}\begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}^T \tag{47-a}$$

$$= \hat{A}_K - \hat{B}_K C_3 X - Y^T B_2 \hat{C}_K - Y^T AX$$

$$\begin{bmatrix} B_{K1} \\ \tilde{B}_{K2} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ 0 & N_3 \end{bmatrix}^{-1} \hat{B}_K \tag{47-b}$$

$$\begin{bmatrix} C_{K1} & \tilde{C}_{K2} \end{bmatrix} = \hat{C}_K \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}^{-T} \tag{47-c}$$

We determine N_1, M_1 through $M_1 N_1^T = I_r - X_1 Y_1$. This is always possible due to (25). The other matrices M_2, M_3, N_2, N_3 can be solved from (47-a) under the condition that (2,2) block of $\hat{A}_K - \hat{B}_K C_3 X - Y^T B_2 \hat{C}_K - Y^T AX$ is nonsingular.

One of a non-descriptor controller is given such as:

$$\begin{aligned} &\tilde{A}_K = A_{K11} - A_{K12}A_{K21}, \tilde{B}_K = B_{K1} - A_{K12}\tilde{B}_{K2} \\ &\tilde{C}_K = C_{K1} - \tilde{C}_{K2}A_{K21}, \tilde{D}_K = \tilde{C}_{K2}\tilde{B}_{K2} \end{aligned} \tag{48}$$

5. NUMERICAL EXAMPLES

The first example is from [8] by which we can confirm that our result is compatible with the existing non-descriptor systems.

Example 1: Consider the three-state unstable plant[1]:

$$\begin{aligned} &\dot{x} = Ax + B_1 w + B_2 u \\ &z_\infty = C_1 x + u \\ &z_2 = C_2 x \\ &y = C_3 x + 2w \\ &A = \begin{bmatrix} 0 & 10 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & -5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &C_1 = [1 \quad 0 \quad 0], C_2 = [0 \quad 1 \quad 1], C_3 = [0 \quad 1 \quad 0] \end{aligned} \tag{42}$$

We transform (42) into (7), which has no direct fed-through terms from control/disturbance input to external/measured output. That is, by adopting extra variables such as $\varsigma = 2w, \xi = u$, (42) can be rewritten as follows:

[1] In the output z_2, we omitted the term "u" in order to meet condition (30).

$$\begin{bmatrix} \boldsymbol{I}_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\psi} = \begin{bmatrix} \boldsymbol{A} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \psi + \begin{bmatrix} \boldsymbol{B}_1 \\ 2 \\ 0 \end{bmatrix} w + \begin{bmatrix} \boldsymbol{B}_2 \\ 0 \\ 1 \end{bmatrix} u \qquad (43\text{-a})$$

$$\boldsymbol{z}_\infty = [\boldsymbol{C}_1 \quad 0 \quad 1]\psi$$

$$\boldsymbol{z}_2 = [\boldsymbol{C}_2 \quad 0 \quad 0]\psi$$

$$y = [\boldsymbol{C}_3 \quad 1 \quad 0]\psi$$

with

$$\psi^T = [\boldsymbol{x}^T \quad \varsigma \quad \xi] \qquad (43\text{-b})$$

By using the computation result of [10], we consider the mixed H_2/ H_∞ control problem as follows:

Minimize $\|T_{wz_2}\|_2$ subject to $\|T_{wz_\infty}\|_\infty < 23.6$

Solving LMIs (22), (24), and (35) with $\gamma = 23.6$ yields 6.46 as best constrained H_2 performance, which is slightly lower than the result of [10]. □

Next, we try to design a mixed H_2/ H_∞ controller by using Theorem 7.

Example 2: Consider the following plant that is given by the descriptor format.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \qquad (44)$$

$$z_\infty = [0 \quad 1 \quad 1]\boldsymbol{x} + w + u$$

$$z_2 = [1 \quad 1 \quad 0]\boldsymbol{x}$$

$$y = [1 \quad 0 \quad 1]\boldsymbol{x} + w$$

By applying Theorem 6, we obtain 1.86 as best constrained H_2 performance on the condition that H_∞ norm is less than 1.77. The solutions of LMIs (22), (24), and (35) are as follows:

$$\boldsymbol{P} = \begin{bmatrix} 2.216 & -1.539 & 1.638 & -.536 & -.470 \\ -1.539 & 3.794 & -2.249 & -.359 & -.793 \\ 1.638 & -2.249 & -3.60\times10^7 & 0 & 0 \\ -.536 & -.359 & 0 & -3.60\times10^7 & 0 \\ -.470 & -.793 & 0 & 0 & -3.60\times10^7 \end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix} 2.156 & .707 & -1.599 & .727 & 8.112\times10^{-5} \\ .707 & .577 & 1.011 & -1.194 & .607 \\ -1.599 & 1.011 & -3.60\times10^7 & 0 & 0 \\ .727 & -1.194 & 0 & -3.60\times10^7 & 0 \\ 8.112\times10^{-5} & .607 & 0 & 0 & -3.60\times10^7 \end{bmatrix}$$

$$\boldsymbol{S} = \begin{bmatrix} 2.138 & -1.111 & -.285 \\ .751 & 1.628 & -.325 \\ -2.498 & 0 & 1.128 \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} .836 & -.617 & -.303 \\ -1.066 & 1.011 & .303 \\ .303 & 0 & 1.443 \end{bmatrix}$$

And the coefficient matrices of the controller (8) become:

$$\boldsymbol{A}_K = \begin{bmatrix} -.974 & -1.036 & .944 & 2.305 & -1.654 \\ .467 & -.566 & 3.162 & -1.771 & .549 \\ -.667 & -1.819 & 5.257 & 2.127 & 11.681 \\ -.901 & .625 & 1.244 & -4.309 & 2.282 \\ 1.856 & 2.753 & -6.759 & .201 & -19.774 \end{bmatrix}$$

$$\boldsymbol{B}_K = \begin{bmatrix} .630 \\ .402 \\ 1.256 \\ .221 \\ -1.803 \end{bmatrix}, \boldsymbol{C}_K^T = \begin{bmatrix} -.598 \\ -.339 \\ 1.331 \\ 1.643 \\ -1.243 \end{bmatrix}, \boldsymbol{D}_K = 0, \boldsymbol{E}_K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, the order of the controller is ostensibly increased by 2, because we augmented the original plant in order to eliminate "D terms" in the performance and plant output.

6. CONCLUSION

We considered the mixed H_2/ H_∞ control of linear descriptor systems. An LMI approach to the synthesis problem (based on a linearizing change of variables) reveals the relationship between the H_2 and H_∞ control problem of the descriptor systems. That is, by introducing extra LMI variables (**V** and **U**) which are determined only by matrix $\boldsymbol{E}$, we could derive the generalized LMI conditions of descriptor systems that involve checking H_2 norm and H_∞ norm conditions of the different input-output channels. A controller computation procedure for the mixed H_2/H_∞ control problem based on the solution of linear matrix inequalities is provided. In this case, we assumed the controller was in descriptor form and had the same dimension as the plant. A Non-descriptor controller, however, can be easily derived by carrying out some numerical transformation in advance such as normalized SVD representations on the plant and control system [12]. Finally we showed two examples to confirm the validity of the proposed LMI conditions for descriptor systems.

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