Discrete-Time Robust $\mathcal{H}_\infty$ Filter Design via Krein Space

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Abstract: A new approach to design of a discrete-time robust $\mathcal{H}_\infty$ filter in finite horizon case is proposed. It is shown that robust $\mathcal{H}_\infty$ filtering problem can be cast into the minimization problem of an indefinite quadratic form, which can be solved by implementing the Kalman filter defined in Krein space. The proposed filter is readily derived by simply augmenting the state space model and has the robustness property against the parameter uncertainties of a given system.

Keywords: indefinite quadratic form, Krein space, robust $\mathcal{H}_\infty$ filter, parameter uncertainty

1. Introduction

The optimal results in the celebrated Wiener and/or Kalman filters are based on the minimization of the L_2 or H_2 norm of the corresponding estimation error. These filters require two critical assumptions: the system model is perfect; the spectral properties about the exogenous noises are exactly known[1], [2]. However these assumptions limit the application of the filters in practice where only approximated system models are available and the statistics of the exogenous signals are not fully known or unavailable. Those limitation recently led many researchers to study $\mathcal{H}_\infty$ filtering algorithm, which is more robust against unknown exogenous noises and less sensitive to parameter variation or uncertainty. In $\mathcal{H}_\infty$ setting, the exogenous signals are assumed to be energy bounded rather than Gaussian. The $\mathcal{H}_\infty$ filter design object is to minimize the $\mathcal{H}_\infty$ norm of the system, which means worst-case estimation error gain to exogenous signals. This gives the $\mathcal{H}_\infty$ filter another name, minimax estimation. In other words, $\mathcal{H}_\infty$ filter guarantees the robustness against all possible disturbances in worst-cases.

There have been several approaches to $\mathcal{H}_\infty$ filter design: ARE[3], [4], [5], [6], interpolation[7], polynomial equation[8], game-theoretic[9], [10] and, more recently, LMI approaches[11], [12]. However, most of the works mentioned above require that the system model is precisely known, apart from the exogenous noises. Even though $\mathcal{H}_\infty$ filter itself is robust against the uncertainties of system model because no assumption on the noises are needed in the design of nominal $\mathcal{H}_\infty$ filter. Those uncertainties are interpreted as exogenous noises, and there exist a limitation to consider the parameter uncertainties of system as exogenous noises. Moreover, it is pointed out that the nominal $\mathcal{H}_\infty$ filter is not robust to the parameter variations of the system[5], [13].

To get improvement of performance, the robustness of $\mathcal{H}_\infty$ filters against the parameter uncertainty is taken into account. Several results have been obtained on the robust $\mathcal{H}_\infty$ filtering [3], [5], [11]. These works deal with so-called norm bounded uncertainty in state and/or in output matrices, and results are obtained by using ARE. The ARE approach for both $\mathcal{H}_\infty$ [3], [5] and $\mathcal{H}_2$ [14], [15] filters involves the conversion of a robust filtering problem into a scaled filtering problem, which transforms the uncertainty into a scaling parameter. Recently, the robust $\mathcal{H}_\infty$ filtering technique for a wider class of parameter uncertainty described by the integral quadratic constraints (IQC's) has been addressed[11], which uses LMI and an $\mathcal{S}$- procedure[16].

In this paper a new robust $\mathcal{H}_\infty$ filter is proposed for the discrete-time system with norm-bounded parametric uncertainties. The uncertainties and exogenous disturbances are described by the energy bound constraint, i.e., sum quadratic constraints (SQC's). It is shown that the SQC's can be converted into an indefinite quadratic cost function to be minimized in an indefinite metric space, known as a Krein space, and it is found that the Kalman filter designed in the Krein space[18] is a solution of the minimization problem. After introducing a Krein space state-space model, which includes the uncertainty, one can easily write a robust version of the Krein space Kalman filter by modifying the measurement matrix and the variance of measurement noises in the original Krein space Kalman filter. Since the resulting robust Kalman filter has the same recursive structure as a conventional Kalman filter has, a robust filtering scheme can be readily designed via the proposed method. Moreover, the condition for a minimum or existence of the solution of $\mathcal{H}_\infty$ problem can also be expressed in terms of quantities easily related to the basic Riccati equations of the Kalman filter. Therefore, it is emphasized that the existing robust $\mathcal{H}_\infty$ filtering problems can be easily interpreted and solved by conventional Kalman filter algorithm. The key idea of applying Krein space estimation theory to robust filtering problem and its application to robust $\mathcal{H}_2$ (Kalman) filter design are represented in our earlier work[19], [20]. Note that only a posteriori filter is presented in this paper. Further results on a priori filter will be given in later work.

2. $\mathcal{H}_\infty$ Filtering Problem

Consider a discrete-time system represented by the following state-space model

$$\left\{\begin{array}{rcl} x_{j+1} & = & F_j x_j + G_j u_j \\ y_j & = & H_j x_j + v_j \\ z_j & = & L_j x_j \end{array}\right. \tag{1}$$

where, $F_j \in \mathcal{C}^{n\times n}, G_j \in \mathcal{C}^{n\times m}, H_j \in \mathcal{C}^{p\times n}, L_j \in \mathcal{C}^{r\times n}$ are given matrices, $x_0 \in \mathcal{C}^n$ is the initial state, $y_j \in \mathcal{C}^p$ is the system output, $z_j \in \mathcal{C}^r$ is an arbitrary linear combination of the states which should be estimated, $u_j \in \mathcal{C}^m, v_j \in \mathcal{C}^p$ are the uncertainty inputs which contain the energy bounded process and measurement noises. Let $\check{z}_{k|k}$ denote the filtered estimate of z_k with given observations $\{y_0, y_1, \cdots, y_k\}$. Then the filtered error is $e_{f,k} = \check{z}_{k|k} - z_k$. if we let $\mathcal{T}$ be a transfer function that maps the uncertain inputs to the estimation errors mentioned above, the $\mathcal{H}_\infty$ norm of $\mathcal{T}$ is defined as the RMS gain of the matrix, that is,

$$\| \mathcal{T} \|_\infty \triangleq \sup_{u\neq 0} \frac{\| \mathcal{T}u \|_{rms}}{\| u \|_{rms}} = \sup_{u\neq 0} \frac{\| \mathcal{T}u \|_2}{\| u \|_2}, \tag{2}$$

where $\| u \|_2$ is the L_2 norm of a sequence u_i, i.e., $\| u \|_2^2 \triangleq \sum_{i=1}^{\infty} u_i^* u_i$. Thus, $\mathcal{H}_\infty$ norm indicates the maximum energy gain from an input to an output. According to the filtered error, the level-γ suboptimal robust $\mathcal{H}_\infty$ filtering problem is given below.

Problem 1: (Level-γ $\mathcal{H}_\infty$ A Posteriori Filtering Problem) For a given scalar $\gamma_f > 0$, find an estimation strategy $\check{z}_{k|k} = \mathcal{F}_f(y_0, y_1, \cdots, y_k)$ that achieves $\| \mathcal{T}_k(\mathcal{F}_f) \|_\infty^2 < \gamma_f^2$. In other words, find a strategy that achieves

$$\sup_{x_0, u\in l_2, v\in l_2} \frac{\sum_{j=0}^{k} e_{f,j}^* e_{f,j}}{\bar{x}^* \Pi_0^{-1} \bar{x} + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k} v_j^* v_j} < \gamma_f^2, \tag{3}$$

where $\bar{x} = x_0 - \hat{x}_0$, and Π_0 is a given positive-definite symmetric matrix which indicates how confident we are about the closeness of the initial guess $\hat{x}_0$ to x_0.

3. Robust $\mathcal{H}_\infty$ Filtering Problem

In this section, $\mathcal{H}_\infty$ filtering scheme is extend to the case that system has certain norm bounded parameter uncertainty uncertainties. The uncertainties are described by SQC, which allows us to have indefinite quadratic form. Moreover, another indefinite quadratic form is derived from $\mathcal{H}_\infty$ norm constraint to check positivity condition. The minimum point of the indefinite quadratic form from the SQC will be found by Krein space robust $\mathcal{H}_\infty$ filter. The robust $\mathcal{H}_\infty$ filters is based on the following uncertain discrete-time system,

$$\begin{cases} x_{j+1} &= F_j x_j + G_j u_j \\ y_j &= H_j x_j + v_j \\ z_j &= L_j x_j \\ s_j &= K_j x_j \end{cases} \tag{4}$$

where every quantity is given in (1) except the given uncertain matrix $K_j \in \mathcal{C}^{q\times n}$ and the uncertainty output $s_j \in \mathcal{C}^q$.

3.1. Problem formulation

Referring to the measurement equation, $y_j = H_j x_j + v_j$, in (4), the inequality in Problem 1 implies that for all nonzero x_0 and $\{u_j, v_j\}_{j=0}^k$

$$\frac{\sum_{j=0}^{k} (\check{z}_{j|j} - z_j)^* (\check{z}_{j|j} - z_j)}{\bar{x}^* \Pi_0^{-1} \bar{x} + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j)} < \gamma_f^2. \tag{5}$$

With assumption that $\hat{x}_0 = 0$, the above inequality of $\mathcal{H}_\infty$ norm constraint with level γ_f gives the following indefinite quadratic form

$$\begin{aligned} J_{f1,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) &= x_0^* \Pi_0^{-1} x_0 \\ + \textstyle\sum_{j=0}^{k} u_j{}^* u_j + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j) \\ - \textstyle\sum_{j=0}^{k} \gamma_f{}^{-2} (\check{z}_{j|j} - z_j)^* (\check{z}_{j|j} - z_j) \end{aligned} \tag{6}$$

which satisfies

$$J_{f1,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) \geq 0. \tag{7}$$

Note that the above quadratic form is a straightforward restatement of (5), and it is always true for all suboptimal $\mathcal{H}_\infty$ a posteriori filters with level γ_f.

Let ϵ be a given positive constant; then the uncertainty of the system in (4) is described by the following energy constraint, i.e., SQC

$$\bar{x}_0^* \Pi_0^{-1} \bar{x}_0 + \sum_{j=0}^{k} \{u_j{}^* u_j + v_j{}^* v_j\} - \frac{1}{\gamma_f^2} \sum_{j=0}^{k} e_{f,j}^* e_{f,j} \leq \epsilon + \sum_{j=0}^{k} \| s_j \|^2 . \tag{8}$$

Also letting $\hat{x}_0 = 0$, we can define another indefinite quadratic form from SQC as follows

$$\begin{aligned} J_{f2,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) &= x_0^* \Pi_0^{-1} x_0 \\ + \textstyle\sum_{j=0}^{k} u_j{}^* u_j + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j) \\ - \gamma_f^{-2} \textstyle\sum_{j=0}^{k} (\check{z}_{k|k} - z_k)^* (\check{z}_{k|k} - z_k) - \sum_{j=0}^{k} \| s_j \|^2 . \end{aligned} \tag{9}$$

Referring to the given system equations in (4), we have the following matrix form of $J_{f2,k}$.

$$J_{f2,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) = x_0^* \Pi_0^{-1} x_0 + \textstyle\sum_{j=0}^{k} u_j{}^* u_j \\ + \textstyle\sum_{j=0}^{N} \left(\begin{bmatrix} y_j \\ 0 \\ \check{z}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} x_j \right)^* R \left(\begin{bmatrix} y_j \\ 0 \\ \check{z}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} x_j \right), \tag{10}$$

where $R = diag(I, -I, -\gamma^{-2} I)$, which is indefinite.

The object is to find the state estimates used to calculate the minimum point of $J_{f2,k}$. Also note that $J_{f2,k}$ equals to $J_{f1,k}$ when the uncertainty is neglected, that is, $J_{f1,k} = J_{f2,k}|_{K=0}$. Since the $\mathcal{H}_\infty$ norm constraint implies quadratic form $J_{f1,k}$ must be positive (see (6)–(7)), the minimum point of $J_{f1,k}$ must be positive, too. We then have the following problem statement.

Problem 2: (Robust A Posteriori $\mathcal{H}_\infty$ Filtering Problem) Given uncertain system (4) with SQC (8), a robust a posteriori $\mathcal{H}_\infty$ filtering problem is to find the estimates used to calculate the minimum point of the indefinite quadratic function $J_{f2,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$. Therefore, the following two conditions must be satisfied.

(a) *Minimum Condition*: $J_{f2,k}$ has a minimum with respect to $\{x_0, u_0, \cdots, u_k\}$.

(b) *Positivity Condition*: The $\{\check{z}_{j|j}\}_{j=0}^k$ can be chosen such that the minimum value of $J_{f2,k}|_{K=0}$, equal to minimum of $J_{f1,k}$, is positive, i.e., $\min J_{f2,k}|_{K=0} = \min J_{f1,k} > 0$.

3.2. Equations of robust a posteriori $\mathcal{H}_\infty$ filter

The solution to Problem 2 is given blow. Note that the solution has the same structure as the $\mathcal{H}_\infty$ filtering solution in

[4], except the augmented matrices to represent uncertainties.

Theorem 1: (Krein Space Robust A Posteriori $\mathcal{H}_\infty$ Filter) For a given system in (4) and $\gamma > 0$, if the $\begin{bmatrix} F_j & G_j \end{bmatrix}$ have full rank, then a suboptimal robust $\mathcal{H}_\infty$ filter that achieves $\| \mathcal{T}_k(\mathcal{F}_f) \|_\infty^2 < \gamma_f^2$ exist if, and only if,

$$P_j^{-1} + H_j^* H_j - K_j^* K_j - \gamma^{-2} L_j^* L_j > 0, \quad j = 0, 1, \cdots, k \quad (11)$$

where $P_0 = \Pi_0$ and P_j satisfies the Riccati recursion

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} P_j F_j^* \quad (12)$$

with

$$R_{e,j} = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^*, \quad (13)$$

where $P_j \triangleq P_{j|j-1}$. In this case, one possible level-γ $\mathcal{H}_\infty$ filter is

$$\check{z}_{j|j} = L_j \hat{x}_{j|j},$$

where the measurement-updated estimate $\hat{x}_{j|j}$ is recursively obtained as

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1} \begin{bmatrix} y_{j+1} - H_{j+1} F_j \hat{x}_{j|j} \\ -K_{j+1} F_j \hat{x}_{j|j} \end{bmatrix}, \quad (14)$$

where

$$K_{s,j+1} = P_{j+1} \begin{bmatrix} H_{j+1} \\ K_{j+1} \end{bmatrix}^* \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_{j+1} \\ K_{j+1} \end{bmatrix} P_{j+1} \begin{bmatrix} H_{j+1} \\ K_{j+1} \end{bmatrix}^* \right)^{-1} \quad (15)$$

An alternative form, i.e., the predicted form of this filter is

$$\begin{aligned} \hat{x}_{j+1|j} &= F_j \hat{x}_{j|j-1} + F_j P_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \\ &\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j \hat{x}_{j|j-1} \\ -K_j \hat{x}_{j|j-1} \end{bmatrix} \\ \check{z}_{j|j} &= L_j \hat{x}_{j|j-1} + L_j P_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \\ &\left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j \hat{x}_{j|j-1} \\ -K_j \hat{x}_{j|j-1} \end{bmatrix} \end{aligned} \quad (16)$$

3.3. Proof of theorem 1

To apply the Krein space projection method in [17] to the deterministic minimization problem given in Problem 2, we can introduce the Krein space state-space equations corresponding to the indefinite quadratic form $J_{2,k}(x_0, u, y)$ for the robust $\mathcal{H}_\infty$ filtering problem as follows

$$\left\{ \begin{aligned} \mathbf{x}_{j+1} &= F_j \mathbf{x}_j + G_j \mathbf{u}_j \\ \begin{bmatrix} \mathbf{y}_j \\ \mathbf{0} \\ \check{\mathbf{z}}_j \end{bmatrix} &= \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} \mathbf{x}_j + \mathbf{v}_j \end{aligned} \right. \quad (17)$$

with the Gramian

$$< \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_k \\ \mathbf{v}_k \end{bmatrix} >= \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & I\delta_{jk} & 0 \\ 0 & 0 & R\delta_{jk} \end{bmatrix}. \quad (18)$$

where $R = diag(I, -I, -\gamma^{-2} I)$. Now, we can derive the equations for a suboptimal robust $\mathcal{H}_\infty$ filter by applying the Krein space Kalman filter equations in [18] to the Krein space system given in (17)–(18). According to Problem 2, the estimates given by the robust $\mathcal{H}_\infty$ filter also satisfy *Minimum Condition* and *Positivity Condition*.

Conditions for a minimum

Comparing the matrices in (17)–(18) with those of the Krein space Kalman filter in [18] yields the following correspondence

$$Q_j \mapsto I, \quad R_j \mapsto diag(I, -I, -\gamma^2 I), \quad H_j \mapsto \begin{bmatrix} H_j & K_j & L_j \end{bmatrix}^T. \quad (19)$$

Considering the above correspondence with the Riccati recursion in [18], we have the following Riccati recursion

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} P_j F_j^*, \quad (20)$$

which is Eq. (12). Since we can use any conditions for a minimum given in [18], the positivity condition for a minimum is selected.

Lemma 1: (Positivity Condition for a Minimum) If $\begin{bmatrix} F_k & G_k \end{bmatrix}$ is full rank for all k, to have

$$\begin{aligned} P_j^{-1} &+ \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^* \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} \\ &= P_j^{-1} + H_j^* H_j - K_j^* K_j - \gamma^{-2} L_j^* L_j > 0, \end{aligned} \quad (21)$$

which is the minimum condition for $J_{2,k}$ and identical to (11).

We can also use the alternative minimum condition given in [18], which lead us to the following.

Lemma 2: (Inertia Condition for a Minimum) The indefinite quadratic form $J_{2,k}$ has a minimum, if and only if,

$$R = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \quad and \quad R_{e,j} = R + \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^*$$

have the same inertia for all $0 < j < k$.

Using a block triangular factorization of $R_{e,j}$, we can also have the following result.

Corollary 1: (Alternative Condition for a Minimum) The condition of Lemma 2 is equivalent to

$$\Psi_y \triangleq I + H_j P_j H_j^* > 0 \quad (22)$$

$$\Psi_s \triangleq -I + K_j (P_j^{-1} + H_j^* H_j)^{-1} K_j^* < 0 \quad (23)$$

$$\Psi_z \triangleq -\gamma^2 I + L_j (P_j^{-1} + H_j^* H_j - K_j^* K_j)^{-1} L_j^* < 0 \quad (24)$$

Construction of the robust $\mathcal{H}_\infty$ a posteriori filter

We still need to find the state estimates that satisfy the *Positivity Condition* in Problem 2. To find the estimates, we begin with the minimum point of $J_{2,k}$. According to [18], the minimum value of $J_{2,k}$ is given by

$$\min J_{2,k} = \sum_{j=0}^{k} e_j^* \; R_{e,j}^{-1} \; e_j.$$

From $R_{e,j}$ in Lemma 2, the minimum value can be expressed by

$$\min J_{2,k} = \sum_{j=0}^{k} E_j^* R_{e,j}^{-1} E_j, \tag{25}$$

where $R_{e,j} = \begin{bmatrix} I+H_jP_jH_j^* & H_jP_jK_j^* & H_jP_jL_j^* \\ K_jP_jH_j^* & -I+K_jP_jK_j^* & K_jP_jL_j^* \\ L_jP_jH_j^* & L_jP_jK_j^* & -\gamma^2 I+L_jP_jL_j^* \end{bmatrix}$ and

$$E_j = \begin{bmatrix} y_j - \hat{y}_{j|j-1} & -K_j\hat{x}_{j|j-1} & \check{z}_{j|j} - \hat{z}_{j|j-1} \end{bmatrix}^T$$

where $\hat{y}_{j|j-1} = H_j\hat{x}_{j|j-1}$, and $\hat{z}_{j|j-1} = L_j\hat{x}_{j|j-1}$.
Using the LDU block triangular factorization of the $R_{e,j}$ we may rewrite the above as

$$\begin{aligned} \min J_{2,k} &= \sum_{j=0}^{k} \bar{E}_j^* \begin{bmatrix} \Psi_y & 0 & 0 \\ 0 & \Psi_s & 0 \\ 0 & 0 & \Psi_z \end{bmatrix}^{-1} \bar{E}_j & (26) \\ &= \sum_{j=0}^{k} (y_j - \hat{y}_{j|j-1})^* \Psi_y^{-1} (y_j - \hat{y}_{j|j-1}) \\ &\quad + \sum_{j=0}^{k} s_{j|j}^* K_j^* \Psi_s^{-1} K_j s_{j|j} \\ &\quad + \sum_{j=0}^{k} (\check{z}_{j|j} - \hat{z}_{j|j})^* \Psi_z^{-1} (\check{z}_{j|j} - \hat{z}_{j|j}), & (27) \end{aligned}$$

where

$$\bar{E}_j = \begin{bmatrix} y_j - \hat{y}_{j|j-1} & -K_js_{j|j} & \check{z}_{j|j} - \hat{z}_{j|j} \end{bmatrix}^T$$

and Ψ_y, Ψ_s and Ψ_z are given in Corollary 1, and we have defined

$$s_{j|j} \triangleq P_jH_j^*(H_jP_jH_j^* + I)^{-1}(y_j - \hat{y}_j) + \hat{x}_j \tag{28}$$

$$\hat{z}_{j|j} \triangleq \hat{z}_{j|j-1} + L_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix}. \tag{29}$$

Now, we have to choose the estimates $\check{z}_{j|j}$ that satisfy the *Positivity Condition*. There can be many choices of $\check{z}_{j|j}$ such that $\min J_{2,k}|_{K=0} = \min J_{1,k} > 0$. Referring to corollary 1, the second summation term and the third summation term in (27) are negative definite. If we neglect the uncertainties, i.e., $K = 0$, the second term is diminished. Moreover, the third term can be diminished by choosing $\check{z}_{j|j}$ equal to $\hat{z}_{j|j}$, that is,

$$\begin{aligned} \check{z}_{j|j} &= \hat{z}_{j|j} = L_j\hat{x}_{j|j} \\ &= L_j\hat{x}_{j|j-1} + L_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \\ &\quad \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix}, \end{aligned} \tag{30}$$

where $x_{j|j-1}$ can be obtained from the Krein space Kalman filter. With the above choice of $\check{z}_{j|j}$,

$$\begin{aligned} &\min J_{2,k}|_{K=0} = \min J_{1,k} \\ &= \sum_{j=0}^{k} (y_j - \hat{y}_{j|j-1})^* \Psi_y^{-1} (y_j - \hat{y}_{j|j-1}) > 0, \end{aligned} \tag{31}$$

which satisfies the *Positivity Condition*.

Using the predicted form of the Krein space Kalman filter in [18] allows us to write

$$\hat{x}_{j+1|j} = F_j\hat{x}_{j|j-1} + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \\ \check{z}_{j|j} - L_j\hat{x}_{j|j-1} \end{bmatrix} \tag{32}$$

$$\check{z}_{j|j} = L_j\hat{x}_{j|j-1} + L_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix}, \tag{33}$$

where $K_{p,j} = F_jP_j \begin{bmatrix} H_j^* & K_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}$. Let

$$\Phi = L_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \tag{34}$$

and substitute the second of the above equations into the first to obtain the followings

$$\begin{aligned} \hat{x}_{j+1|j} &= F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \\ \Phi \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \end{bmatrix} \\ &= F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} H_j \\ K_j \\ L_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} I & 0 \\ \Phi & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \\ 0 \end{bmatrix} \end{aligned} \tag{35}$$

The block triangular factorization of $R_{e,j}$, which can be obtained by using the formulas in Appendix, is given by

$$R_{e,j}^{-1} = \begin{bmatrix} 1 & -\Psi_y^{-1}H_jP_jK_j^* & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Psi_y^{-1} & 0 & 0 \\ 0 & \Psi_s^{-1} & 0 \\ 0 & 0 & \Psi_z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -K_jP_jH_j^*\Psi_y^{-1} & I & 0 \\ \Delta_l & \Delta_s & I \end{bmatrix} \tag{36}$$

where Ψ_y, Ψ_s and Ψ_z are given in corollary 1; Δ_l and Δ_s can be obtained through Appendix; $\times$ denotes irrelevant entries. In addition, using 2×2 block triangular factorization, we can find that Φ in (34) has the following relation

$$\Phi = \begin{bmatrix} -\Delta_l & -\Delta_s \end{bmatrix}. \tag{37}$$

With the above relation, replacing the result for $R_{e,j}^{-1}$ into (35) and some tedious works allow us to rewrite the recursion

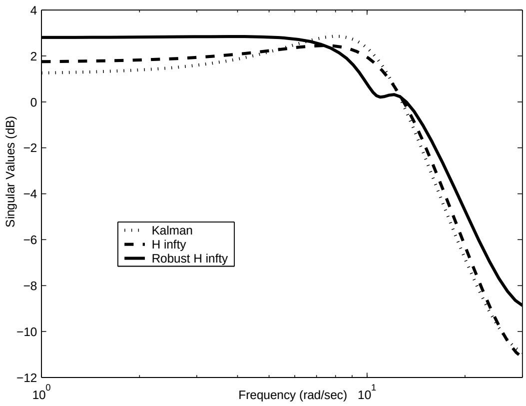

Fig. 1. Singular value plot : nominal case($\Delta = 0$)

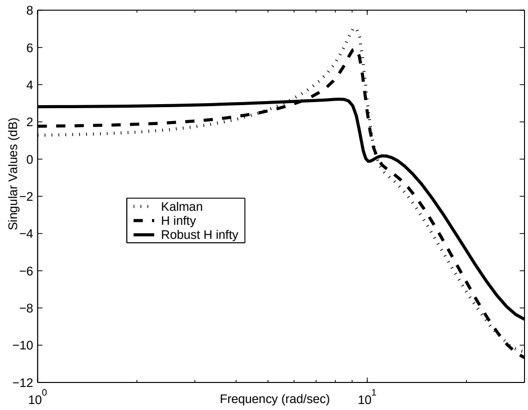

Fig. 2. Singular value plot : uncertain case($\Delta = 1$)

for $\hat{x}_j$ as

$$\hat{x}_{j+1|j} = F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} H_j^*\Psi_y + (-H_j^*\Psi_yH_jP_jK_j^* + K_j^*)\Psi_s^{-1}(-K_jP_jH_j^*\Psi_y) \\ (-H_j^*\Psi_yH_jP_jK_j + K_j^*)\Psi_s^{-1} \\ \times \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \\ 0 \end{bmatrix} \quad (38)$$

$$= F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \quad (39)$$

Therefore, the recursion in (33) can be rewritten as

$$\hat{x}_{j+1|j} = F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \quad (40)$$

$$\check{z}_{j|j} = L_j\hat{x}_{j|j-1} + L_jP_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix} \quad (41)$$

which is equal to (16).

Since $\check{z}_{j|j} = \hat{z}_{j|j} = L_j\hat{x}_{j|j}$, from the second equation of (33), we can derive the following definition

$$\hat{x}_{j|j} \triangleq \hat{x}_{j|j-1} + P_j \begin{bmatrix} H_j^* & K_j^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_j \\ K_j \end{bmatrix} P_j \begin{bmatrix} H_j \\ K_j \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_j - H_j\hat{x}_{j|j-1} \\ -K_j\hat{x}_{j|j-1} \end{bmatrix}. \quad (42)$$

Increasing all of the time index in (42) and using $\hat{x}_{j+1|j} = F_j\hat{x}_{j|j}$, a recursion for $\hat{x}_{j|j}$ is readily obtained as

$$\hat{x}_{j+1|j+1} = \hat{x}_{j+1|j} + P_{j+1} \begin{bmatrix} H_{j+1}^* & K_{j+1}^* \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} H_{j+1} \\ K_{j+1} \end{bmatrix} P_{j+1} \begin{bmatrix} H_{j+1} \\ K_{j+1} \end{bmatrix}^* \right)^{-1} \begin{bmatrix} y_{j+1} - H_{j+1}F_j\hat{x}_{j|j} \\ -K_{j+1}F_j\hat{x}_{j|j} \end{bmatrix} \quad (43)$$

which is equal to (14).

4. Numerical Example

In order to demonstrate the properties of the proposed $\mathcal{H}_\infty$ filter, consider the following discrete-time system

$$x_{j+1} = \begin{bmatrix} 0.5079 & 0.7594 \\ -0.7594 & 0.2801 \end{bmatrix} x_j + \begin{bmatrix} 0 \\ 0.3261 \end{bmatrix} \Delta_j \begin{bmatrix} 0 & 0.92 \end{bmatrix} + \begin{bmatrix} 0.4921 \\ 0.7594 \end{bmatrix} u_j \quad (44)$$

with velocity measurement described by

$$y_j = \begin{bmatrix} 0 & 1 \end{bmatrix} x_j + v_j \quad (45)$$

where it is desired to estimate

$$z_j = \begin{bmatrix} 1 & 0 \end{bmatrix} x_j \quad (46)$$

and it is also assumed that the exogenous noises are energy bounded signals. The parameter uncertainty Δ_j satisfies $\| \Delta_j \| \leq 1$.

The performance of the nominal Kalman, the nominal $\mathcal{H}_\infty$ and the proposed robust $\mathcal{H}_\infty$ filters is compared by the singular value plot of their error systems in steady-state. For a given $\gamma = 1.5$, the simulation results are shown in Fig.1 and Fig.2 for the nominal($\Delta_j = 0$) and the uncertain($\Delta_j = 1$) cases, respectively. It is shown that the peak magnitude of energy gain of the proposed robust $\mathcal{H}_\infty$ filter is lower than those of other nominal filters in worst-case. Of course, the nominal performance of the proposed robust $\mathcal{H}_\infty$ filter is not as good as the other filters. In the uncertain case, however, its robustness is achieved at the cost of compromising $\mathcal{H}_\infty$ performance while the nominal $\mathcal{H}_\infty$ and the nominal Kalman filters are relatively sensitive to the parameter variations of the system. From the results, it can be concluded that the proposed robust $\mathcal{H}_\infty$ filter guarantees the robustness against all available disturbances in the presence of parameter uncertainty but it may be overly conservative in mean square error sense due to considering whole uncertain situations.

5. Conclusion

Kalman filter solution obtained in Krein space intuitively lead us to evaluate the solution to robust $\mathcal{H}_\infty$ filtering problem. Suboptimal $\mathcal{H}_\infty$ constraint and parameter uncertainty can be combined into SQC, which is easily converted into an indefinite quadratic form. Minimizing solution of this quadratic form give us the estimates that satisfy the $\mathcal{H}_\infty$ constraint. Thus, we have the solution to robust $\mathcal{H}_\infty$ filtering problem with the simple way compared with the existing solutions. Although $\mathcal{H}_\infty$ filters have inherent robustness against uncertainties, the proposed robust $\mathcal{H}_\infty$ filter shows better robustness against parameter uncertainties than nominal $\mathcal{H}_\infty$ filter and conventional Kalman filter.

Appendix

Block triangular factorization is used frequently in this paper. Although 2×2 triangular factorization formulas are well known and found in many text books, we need 3 ×3 block triangular factorization because we have augmented matrices including uncertain matrix. Say that we have the following 3×3 block matrix.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}^{-1} = \begin{bmatrix} I & X_1 & X_2 \\ 0 & I & X_3 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ X_4 & I & 0 \\ X_5 & X_6 & I \end{bmatrix} \tag{47}$$

After some Gaussian elimination work, the following results are obtained.

$$X_1 = -A^{-1}B \tag{48}$$
$$X_2 = A^{-1}B(E - DA^{-1}B)^{-1}(F - DA^{-1}C) - A^{-1}C \tag{49}$$
$$X_3 = -(E - DA^{-1}B)^{-1}(F - DA^{-1}C) \tag{50}$$
$$X_4 = -DA^{-1} \tag{51}$$
$$X_5 = (H - GA^{-1}B)(E - DA^{-1}B)^{-1}DA^{-1} - GA^{-1} \tag{52}$$
$$X_6 = -(H - GA^{-1}B)(E - DA^{-1}B)^{-1} \tag{53}$$
$$D_1 = A^{-1} \tag{54}$$
$$D_2 = (E - DA^{-1}B)^{-1} \tag{55}$$
$$D_3 = (J - GA^{-1}C - (H - GA^{-1}B)(E - DA^{-1}B)^{-1} (F - DA^{-1}C))^{-1}. \tag{56}$$

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