Dual-rate Digital Controller Design for Continuous-time Linear Systems

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Abstract: The lifting technique is a standard control procedure that is commonly applied to dual-rate systems, where a critical difficulty is that care must be taken so that the resulting equivalent system preserves the causality constraint between the control signal and the measured output. To overcome this difficulty, the most attractive result has been suggested by defining control time sequences as the union of sample and hold time sequences. However, the sacrifice of regular control period scheme results in some serious disadvantages; restrictions on the implementation to hardware and the corresponding inefficient control scheme. On the contrary, this paper proposes a novel dual-rate control technique, which redescribes the system as a control-rate-based system having regular control period and designs the controller, with no causality constraint, through *Linear Matrix Inequality* (LMI) formulation.

Keywords: dual-rate, multi-rate, LMI

1. Introduction

For a sampled-data control system, the plant is in general a continuous-time LTI system and the controller is composed of A/D converters(samplers), a digital computer, and D/A converters(holds). Hence, a sampled-data control system is a hybrid system involving both continuous-time and discrete-time signals. In many applications, the samplers and the holds do not necessarily operate in the same rate. In such cases, the system is called a *multi-rate* control system [1].
Analysis and synthesis of such systems have been of continuing interest for several decades. A number of important results have been reported in the literature; $\mathcal{H}_\in$ [1], [2], $\mathcal{H}_\infty$ [2], [3], [4], [5], LQG problem [5], [6] and nonlinear problem [7]. One of the standard techniques for modeling of the multi-rate system is *blocking* or *lifting*. It is mainly based on enlarging the number of components of the input and/or output vector and transforming the model into a single-rate one.
But, the main difficulty encountered in the lifting approach is the causality constraint imposed on the controller, which has limited the applicability of standard state-space solutions on the lifted system description. While the lifted system is a single-rate system, its controller realization should satisfy certain structural constraints that arise directly from the causal nature of the original system and the nature of the lifting operation. A recent paper [8] made excellent efforts to overcome such constraints. Contrary to the previous works, this paper defines the control time sequences as the union of the sample and hold time sequences, such that the formulation can be represented as a set of LMIs with no causality constraint. However, the results have some disadvantages; the irregular control period, which imposes serious restrictions on the implementation to hardware, and the inefficient control scheme, which happens when the slow hold does not transmit the control changes carried out by the fast sampler into the plant.
In this paper, we propose a stabilizer of *dual-rate* systems, an interesting particular case of multi-rate systems, where separately synchronized sample and hold operations are performed with different rates. A control-rate-based switching control scheme will be addressed, where regular control periods are accomplished and, much better, the unnecessary control transitions occurring in [8] are removed. The assumption that the sampling does not occur at exactly the same as the hold operation, a hard constraint needed in [8], is also eliminated in our methodology. Both the plant states and the one-step past input are augmented to synchronize with the controller recursion rate, permitting controller gain calculation in the sense of using maximum available information. Augmented output dimensions vary periodically to allow a controller to use all available data for each input calculation. In fact, the proposed controller is a periodic system; not only the gains but also the dimensions of the controllers are periodic. Its structure needs no causality constraint which is an essential condition for the lifting approach.

This paper is organized as follows. The system models and some preliminaries on dual-rate systems are stated in Section 2. The controller design problem and the stability issue are then presented in Section 3. Finally, a numerical example will be addressed in Section 4.

Notations : The following symbols are used throughout the paper.

$\mathbf{R}^{\mathbf{m}\times\mathbf{n}}$	m by n matrices with real components
$\mathbf{R}^+$	the set of positive real numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers
τ_s	the sample period
$1/\tau_s$	the sample rate

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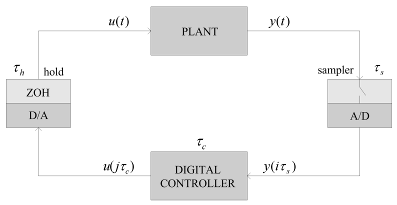

Fig. 1. digital control system structure

τ_h	the hold period
τ_c	the control period
$1/\tau_c$	the control rate

2. System Models

2.1. Digital control systems structure

Figure 1 shows the topology of the typical digital control system [9]. Let us consider first the action of the *analog to digital*(A/D) converter on a signal. This device acts on a physical variable, most commonly an electrical voltage, and converts it into a binary number. The conversion from the analog signal $y(t)$ occurs repetitively at instants of time that are τ_s apart. τ_s is called the *sample period* and $1/\tau_s$ is the *sample rate*. The sampled signal is $y(i\tau_s)$, where i can be any integer valve. The digital controller operates on samples $y(i\tau_s)$ of the plant output and the dynamics are implemented by algebraic recursive equations called difference equations. The result of the difference equation is a discrete $u(j\tau_c)$ at each recursion instant. This signal is converted to a continuous $u(t)$ by the D/A and hold. The D/A converts the binary number to an analog signal, and a *zero order hold*(ZOH) maintains that same voltage throughout the *hold period* τ_h. The resulting $u(t)$ is then applied to the actuator in precisely the same manner as the continuous implementation. We make the assumption here that the sample/hold periods are fixed.

2.2. Controller recursion rate selection

The digital controller is usually accomplished in a digital computer. And for given hardware restrictions of sample/hold periods, the selection of the best *control period* τ_c, or *control rate* $1/\tau_c$, is a compromise among many factors.

Whereas a decrease in control rates yields more time for control calculations and thus more capability available from a given computer, an increase in control rates provides better tracking performance and less sensitivity to disturbances. Because of the recent development of CPU, it is possible to set the control recursion interval τ_c short enough to obtain satisfactory performance without deteriorating calculation capability. Factors that provide an upper limit on effective control rates is the hold period τ_h of the ZOH. The ZOH does not transmit changes of the discrete signal $u(j\tau_c)$ within the hold period τ_h into the plant.

In this paper, the digital control systems that have the control recursion interval $\tau_c = \tau_h$ are considered. No comparative assumptions for sample/hold periods are in need.

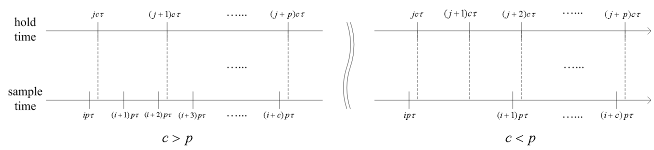

Fig. 2. the timing diagram of dual-rate systems

2.3. Structural properties in state space descriptions of the dual-rate system

Consider the continuous linear time invariant plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x \in \mathbf{R^n}$ is the state variable, $y \in \mathbf{R^l}$ is the output signal and $u \in \mathbf{R^m}$ is the input vector. From now on, without loss of generality, we assume that the hold period τ_h is $c\tau$ and the sample period τ_s is $p\tau$, where $c \in \mathbb{Z}^+$ and $p \in \mathbb{Z}^+$ having a coprime relation and $\tau \in \mathbf{R}^+$. To avoid symbolic confusion, we use *parenthesis* for the notation of continuous time signal and *bracket* for the discrete one.

By using the state space description, we can obtain the difference equations that represent the behaviour of the continuous plant. For example, the discrete time state discretized by the sampler at time $(i+3)p\tau$ in Figure 2 becomes

$$\begin{aligned} x[(i+3)p\tau] &= e^{Ap\tau}x[(i+2)p\tau] \\ &+ \left[\int_{(i+2)p\tau}^{(j+1)c\tau} e^{A\{(j+3)p\tau-t\}}Bdt\right]u[jc\tau] \\ &+ \left[\int_{(j+1)c\tau}^{(i+3)p\tau} e^{A\{(i+3)p\tau-t\}}Bdt\right]u[(j+1)c\tau]. \end{aligned} \tag{2}$$

The abnormal structure of this state space form is that plural inputs are applied to a single-step state recursion process. To deal with this situation in a simplified way, the interval-wise periodicity has been used; although the sample/hold periods are not the same, the whole system is periodic. The lifting approach or the block multi-rate input-output model approach [10] is the most common method.

3. Control-rate-based Controller Design

In this section, a new control scheme for dual-rate systems is proposed. This scheme starts with an observation that, from the state space descriptions of the dual-rate system (2), we know that the one-step past input behaves like a state in the state recursion equation. Therefore, differently from the previous works, we shall use the one-step past input in company with the plant outputs for the controller gain calculation. Surely, the past input is an available data; using a memory block in the computer, it can be stored and reused for the next gain computation.

3.1. A novel description of dual-rate systems

In Figure 3, all the sample/hold instances are labeled with λ's and N_i indicates the number of sample operation for i-th control block. Now, we have p control blocks for each $pc\tau$-time interval, and we shall augment states and input in each

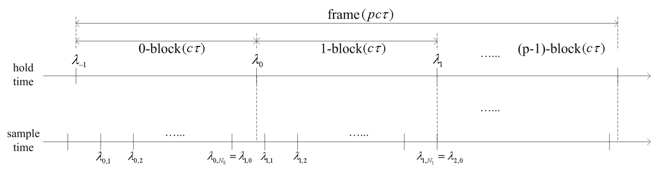

Fig. 3. the timing diagram of dual-rate system

block such that a periodic system with period p will be used as our objective state space equation. The augmented states and input would be

$$\tilde{x}[i] \triangleq \begin{bmatrix} x[\lambda_{i,1}] \\ x[\lambda_{i,2}] \\ \vdots \\ x[\lambda_{i,N_i}] \\ u[\lambda_{i-1}] \end{bmatrix}, \quad u[i] \triangleq u[\lambda_i], \tag{3}$$

and the corresponding plant would be

$$\begin{aligned} \tilde{x}[i+1] &= \tilde{A}_i\tilde{x}[i] + \tilde{B}_i u[i], \\ \tilde{y}[i] &= \tilde{C}_i\tilde{x}[i], \end{aligned} \tag{4}$$

where, system matrices are periodic such that $\tilde{A}_i = \tilde{A}_{i+rp}$, $\tilde{B}_i = \tilde{B}_{i+rp}$ and $\tilde{C}_i = \tilde{C}_{i+rp}$,$\forall r \in \mathbb{Z}$ and

$$\tilde{A}_i \triangleq \begin{bmatrix} 0 & \cdots & 0 & A_{i,1} & B^1_{i,1} \\ 0 & \cdots & 0 & A_{i,2} & B^1_{i,2} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & A_{i,N_i} & B^1_{i,N_i} \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \tilde{B}_i \triangleq \begin{bmatrix} B^2_{i,1} \\ B^2_{i,2} \\ \vdots \\ B^2_{i,N_i} \\ I \end{bmatrix},$$

$$\tilde{C}_i \triangleq \begin{bmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & C & 0 \\ 0 & \cdots & \cdots & 0 & I \end{bmatrix}, \tag{5}$$

$$A_{i,j} \triangleq e^{A(\lambda_{i+1,j}-\lambda_{i+1,0})}, \tag{6}$$

$$B^1_{i,j} \triangleq \int_{\lambda_{i+1,0}}^{\lambda_{i+1}} e^{A(\lambda_{i+1,j}-t)}Bdt, \tag{7}$$

$$B^2_{i,j} \triangleq \int_{\lambda_{i+1}}^{\lambda_{i+1,j}} e^{A(\lambda_{i+1,j}-t)}Bdt, \tag{8}$$

for $i = 0, \cdots, p-1$.

Remark 1 : We know that the system description (4) reflects an aspect of periodic systems, linear difference equations with periodic coefficients. Depending on the number of the sample/hold operations in each control block, dimensions of the signal vectors and the corresponding system matrices vary with period p.

Remark 2 : The augmented system (4) does not have any modification on input $u[i]$; it is equivalent to the original system input, showing the possibility of the control-rate-based controller design.

Remark 3 : As a direct result of the sampling operations, the signal aliasing issue should be considered; the original signal must be recovered unambiguously by their samples. The concerning theorem was developed by H. Nyquist: One can recover a signal from its samples if the sampling frequency is at least twice the highest frequency(the *Nyquist frequency*) in the signal. This is at the least condition for the control scheme based on the sampling operation and we shall assume that the sampler operates beyond the Nyquist frequency.

Remark 4 : Using the one-step past input for the controller gain calculation is an essential idea of our scheme, and the corresponding diagonal term "I" in the matrix $\tilde{C}_i$ reflects this structural feature obviously. In comparison with our descriptions, [8] uses only the plant outputs for the controller gain calculation so that the matrix $\tilde{C}_2(t_k)$ in [8] obeys the conventional output matrix form.

In this manner, we can achieve the conventional control scheme with maximal information use. No special causality constraint or sample/hold time synchronization is required.

3.2. Controller Synthesis and stability analysis

Now, we can consider a dynamic output feedback controller, which switches itself depending on current plant system matrices, as

$$\begin{aligned} x_c[i+1] &= F_i x_c[i] + G_i\tilde{y}[i], \\ u[i] &= H_i x_c[i] + J_i\tilde{y}[i], \end{aligned} \tag{9}$$

where, $F_i = F_{i+rp}$, $G_i = G_{i+rp}$, $H_i = H_{i+rp}$ and $J_i = J_{i+rp}$, $\forall\ r \in \mathbb{Z}$. We shall analyze the stabilization of the closed loop system:

$$\bar{x}[k+1] = (\bar{A}_k + \bar{B}_k\Sigma_k\bar{C}_k)\bar{x}[k], \tag{10}$$

where

$$\bar{A}_k \triangleq \begin{bmatrix} \tilde{A}_k & 0 \\ 0 & 0 \end{bmatrix}, \bar{B}_k \triangleq \begin{bmatrix} 0 & \tilde{B}_k \\ I & 0 \end{bmatrix}, \bar{C}_k \triangleq \begin{bmatrix} 0 & I \\ \tilde{C}_k & 0 \end{bmatrix},$$

$$\Sigma_k \triangleq \begin{bmatrix} F_k & G_k \\ H_k & J_k \end{bmatrix}, \bar{x}[k] \triangleq \begin{bmatrix} \tilde{x}[k] \\ x_c[k] \end{bmatrix}. \tag{11}$$

Theorem 1: The closed loop control system (10) is asymptotically stabilizable if there exist positive definite matrices X_k and $\bar{X}_k$ with period p satisfying the following conditions

$$\tilde{B}_k^{\perp T}(\tilde{A}_k\bar{X}_k\tilde{A}_k^T - \bar{X}_{k+1})\tilde{B}_k^{\perp} < 0, \tag{12}$$

$$\tilde{C}_k^{T\perp T}(\tilde{A}_k^T X_{k+1}\tilde{A}_k - X_k)\tilde{C}_k^{T\perp} < 0, \tag{13}$$

$$\begin{bmatrix} X_k & I \\ I & \bar{X}_k \end{bmatrix} \geq 0, \tag{14}$$

for all $k = 0, \cdots, p-1$, where $\tilde{B}_k^{\perp}$ and $\tilde{C}_k^{T\perp}$ are any matrices of maximum rank such that $\tilde{B}_k^{\perp T}\tilde{B}_k = 0$ and $\tilde{C}_k^{T\perp T}\tilde{C}_k^T = 0$, respectively.

Proof : Consider a Lyapunov function associated with positive definite matrices such as,

$$V(\bar{x}[k], k) = \bar{x}[k]^T P_k\bar{x}[k], \tag{15}$$

where P_k is a positive definite matrix. Then, the corresponding stability condition is

$$(\bar{A}_k + \bar{B}_k\Sigma_k\bar{C}_k)^T P_{k+1}(\bar{A}_k + \bar{B}_k\Sigma_k\bar{C}_k) - P_k < 0. \quad (16)$$

By applying the Schur complement technique [12], we obtain the inequality

$$Q_{k,k+1} + U_{k,k+1}\Sigma_k V_k^T + V_k\Sigma_k^T U_{k,k+1}^T < 0, \quad (17)$$

where

$$Q_{k,k+1} \triangleq \begin{bmatrix} -P_{k+1} & P_{k+1}\bar{A}_k \\ \bar{A}_k^T P_{k+1} & -P_k \end{bmatrix},$$

$$U_{k,k+1} \triangleq \begin{bmatrix} P_{k+1}\bar{B}_k \\ 0 \end{bmatrix}, \; V_k \triangleq \begin{bmatrix} 0 \\ \bar{C}_k^T \end{bmatrix}. \quad (18)$$

To proceed, we partition matrix P_k as

$$P_k \triangleq \begin{bmatrix} X_k & Y_k \\ Y_k^T & Z_k \end{bmatrix}, P_k^{-1} \triangleq \begin{bmatrix} \bar{X}_k & \bar{Y}_k \\ \bar{Y}_k^T & \bar{Z}_k \end{bmatrix}. \quad (19)$$

Let the orthogonal complements of $U_{k,k+1}$ and V_k be

$$U_{k,k+1}^\perp = \begin{bmatrix} 0 & P_{k+1}^{-1}\bar{B}_k^\perp \\ I & 0 \end{bmatrix}, V_k^\perp = \begin{bmatrix} I & 0 \\ 0 & \bar{C}_k^{T\perp} \end{bmatrix}. \quad (20)$$

After applying the elimination lemma [12] and by letting

$$\bar{B}_k^\perp = \begin{bmatrix} \tilde{B}_k^\perp \\ 0 \end{bmatrix}, \bar{C}_k^{T\perp} = \begin{bmatrix} \tilde{C}_k^{T\perp} \\ 0 \end{bmatrix} \quad (21)$$

we get two matrix inequalities (12) and (13). For condition $P_k > 0$, we add the condition (14) [13].

Remark 5 : Using any feasible matrices X_k and $\bar{X}_k$, we can construct P_k as

$$P_k = \begin{bmatrix} X_k & Y_k \\ Y_k^T & Z_k \end{bmatrix} > 0, \quad (22)$$

where $X_k - \bar{X}_k^{-1} = Y_k Z_k^{-1} Y_k^T$. And the corresponding Σ_k of the equation (10) can be accomplished using standard optimization programming techniques and any such solution provides the state space realization for a feasible controller.

Remark 6 : The transformed system description (4) is a standard state-space form revealing itself the causal nature such that no special causality constraint is needed, which is an essential part in the lifting approach. Naturally, the corresponding controller (9) can be obtained by the standard controller synthesis methods, which uses maximum available information for the control gain calculation.

Remark 7 : The resultant formulation was represented as LMIs, which enables one to use the efficient LMI tools and to simplify its computation within the convex optimization framework.

Remark 8 : [8] represents the multi-rate system as a fixed-dimensional state space description by defining the time sequences as the union of the sample and hold time sequences. Its control duration is time varying and the total number of periodically varying controllers necessary to stabilize the original system is $p + c$. However, in our system description, the dimension of the state varies periodically and the controller duration is fixed with $c\tau$, which is more natural selection for the controller design problem. And the number of the switching controllers for the stabilization is p, much less than that in [8].

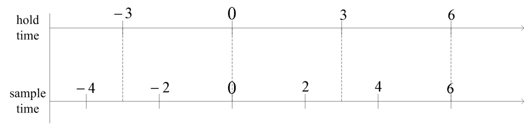

Fig. 4. timing diagram for numerical example

4. Numerical Example

This section presents an example to illustrate the performance of the proposed stabilization scheme. Consider a continuous linear time invariant plant model of (1). The following system matrices

$$A = \begin{bmatrix} -0.5 & -2 \\ 0 & 1.5 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}, C = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad (23)$$

and an initial condition $x(0) = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$ will be used. We put the initial states with negative time indices as zero. We take rate parameters as

$$p = 2, c = 3, \tau = 1. \quad (24)$$

For this case, the corresponding timing diagram will be shown in Figure 4 and the augmented state transition would be repeated as

$$\cdots \rightarrow \overbrace{\begin{bmatrix} x(2) \\ u(0) \end{bmatrix}}^{\tilde{x}(0)} \rightarrow \overbrace{\begin{bmatrix} x(4) \\ x(6) \\ u(3) \end{bmatrix}}^{\tilde{x}(1)} \rightarrow \overbrace{\begin{bmatrix} x(8) \\ u(6) \end{bmatrix}}^{\tilde{x}(2)} \rightarrow \cdots. \quad (25)$$

The periodically varying matrices (5), (6), (7) and (8) can be calculated as

$$\tilde{A}_{2k} = \begin{bmatrix} A_{2k,1} & B_{2k,1}^1 \\ A_{2k,2} & B_{2k,2}^1 \\ 0 & 0 \end{bmatrix}, \tilde{A}_{2k+1} = \begin{bmatrix} 0 & A_{2k+1,1} & B_{2k+1,1}^1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

$$\tilde{B}_{2k} = \begin{bmatrix} B_{2k,1}^2 \\ B_{2k,2}^2 \\ I \end{bmatrix}, \tilde{B}_{2k+1} = \begin{bmatrix} B_{2k+1,1}^2 \\ I \end{bmatrix}, \quad (27)$$

$$\tilde{C}_{2k} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \tilde{C}_{2k+1} = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{bmatrix}, \quad (28)$$

$$A_{2k,1} = e^{2A}, \; B_{2k,1}^1 = \int_1^2 e^{At}Bdt, \; B_{2k,1}^2 = \int_0^1 e^{At}Bdt, \quad (29)$$

$$A_{2k,2} = e^{4A}, \; B_{2k,2}^1 = \int_3^4 e^{At}Bdt, \; B_{2k,2}^2 = \int_0^3 e^{At}Bdt, \quad (30)$$

$$A_{2k+1,1} = e^{2A}, \; B_{2k+1,1}^1 = 0, \; B_{2k+1,1}^2 = \int_0^2 e^{At}Bdt, \quad (31)$$

where $k \in \mathbb{Z}$. The periodic control gains are calculated with the LMI toolbox in the Matlab 5.3, and simulation is performed with the SIMULINK 3.0.1. Figure 5, 6 and 7 illustrate the trajectories of states, input and output, respectively.

5. Conclusions

A new approach for handling dual-rate control situations has been presented. This novel description of the system, synchronized with the control recursion rate, alters not only the system matrices but also the dimensions of the signal vectors periodically. By augmenting the one-step past input to the output vector, it gets rid of the effort in dealing with the causality issue and enables the control scheme in the sense of maximum information use, and even better, non-constrained standard state space form is achieved. The controller duration is fixed with $c\tau$, which brings the hardware implementation problem in [8] to a settlement. The proposed controller is also a periodic dynamic output feedback controller, which switches itself depending on the time varying system matrices.

As a future work, we conceive three kinds of topics; performance problems, nonlinearity analysis related to quantization and its extension to multi-rate systems.

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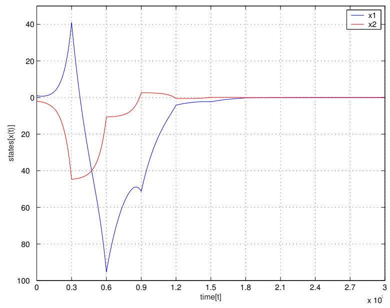

Fig. 5. trajectories of states

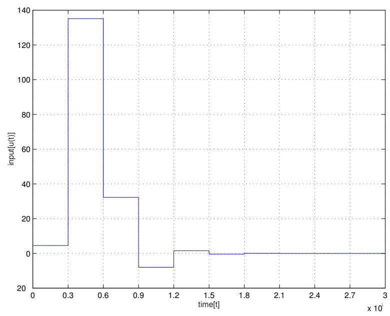

Fig. 6. trajectory of input

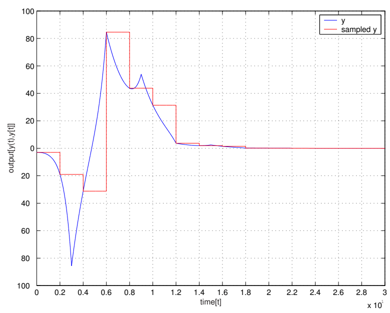

Fig. 7. trajectory of output