An Optimal Structure of Finite-Word-Length Controller Problems in Two Degrees of Freedom Against Colored Noise

Keiji Ohkubo* and Kazuma Miyazawa*

*Department of Electrical and Electronic Engineering, Shinshu University, Nagano, Japan
(Tel:+81-26-269-5183, E-mail:ohkubok@gipwc.shinshu-u.ac.jp)

Abstract: As for control systems, many researchers give optimal structures of the finite-word-length compensator. D. Williamson solved a fixed-point case against colored noise for the LQG problem. Recently, one of the authors derived an optimal filter against colored noise. And consequently, we apply the result to a two-degree-of-freedom control system in this paper. In addition the perturbation of the coefficients is considered. Furthermore, simulation results indicate this method gives better than other structures.

Key-Words: - Fixed-point controller, Optimal solution, Two-degree-of freedom control system, Colored noise, Finite word length

1. INTRODUCTION

It is important to consider the implementation of a fixed-point finite word length (FWL) compensator instead of float-point of a compensator when we want a reasonable-price compensator, or an integrated-circuit compensator. Concerning quantization error, namely round-off errors and coefficient errors, there is a great deal of research for digital filters for example [4]. As for control systems, many researchers have given optimal structures of the compensator as in [2]. However, as in the previous paper, they should assume the noise is white. D. Williamson[1] solved this case against colored noise in the LQG controller. Recently, one of the authors derived an optimal filter against colored noise[3]. And consequently, we applied the result to a two-degree-of-freedom control system in this paper. In addition the perturbation of the coefficients is considered. Furthermore, simulation results indicate this method gives better than other structures.

2. STATE-SPACE REPRESENTATION OF THE DIGITAL CONTROL SYSTEM

Let state-space representations of infinite-word-length (IWL) compensators C_1 and C_2 be

$$C_1 \quad \left\{ \begin{array}{l} \boldsymbol{x}_{c1}[k+1] = \boldsymbol{A}_{c1}\boldsymbol{x}_{c1}[k] + \boldsymbol{b}_{c1}r[k] \\ y_{c1}[k] = \boldsymbol{c}_{c1}\boldsymbol{x}_{c1}[k] + d_{c1}r[k] \end{array} \right. \tag{1}$$

$$C_2 \quad \left\{ \begin{array}{l} \boldsymbol{x}_{c2}[k+1] = \boldsymbol{A}_{c2}\boldsymbol{x}_{c2}[k] + \boldsymbol{b}_{c2}u_c[k] \\ y_{c2}[k] = \boldsymbol{c}_{c2}\boldsymbol{x}_{c2}[k] + d_{c2}u_c[k] \end{array} \right. \tag{2}$$

where $\boldsymbol{x}_{c1}[k] \in R^{n_1}$, $y_{c1}[k] \in R$, $\boldsymbol{A}_{c1} \in R^{n_1 \times n_1}$, $\boldsymbol{b}_{c1} \in R^{n_1 \times 1}$, $\boldsymbol{c}_{c1} \in R^{1 \times n_1}$, $d_{c1} \in R^{1\times 1}$, $\boldsymbol{x}_{c2}[k] \in R^{n_2}$, $y_{c2}[k] \in R$, $\boldsymbol{A}_{c2} \in R^{n_2 \times n_2}$, $\boldsymbol{b}_{c2} \in R^{n_2 \times 1}$, $\boldsymbol{c}_{c2} \in R^{1 \times n_2}$, and $d_{c2} \in R^{1\times 1}$. Also, let the state-space representation of the plant be

$$P \left\{ \begin{array}{l} \boldsymbol{x}_p[k+1] = \boldsymbol{A}_p\boldsymbol{x}_p[k] + \boldsymbol{b}_p u_p[k] \\ y_p[k] = \boldsymbol{c}_p\boldsymbol{x}_c[k] \end{array} \right. \tag{3}$$

where $\boldsymbol{x}_p[k] \in R^m$, $y_p[k] \in R$, $\boldsymbol{A}_p \in R^{m \times m}$, $\boldsymbol{b}_p \in R^{m\times 1}$, and $\boldsymbol{c}_p \in R^{1\times m}$.

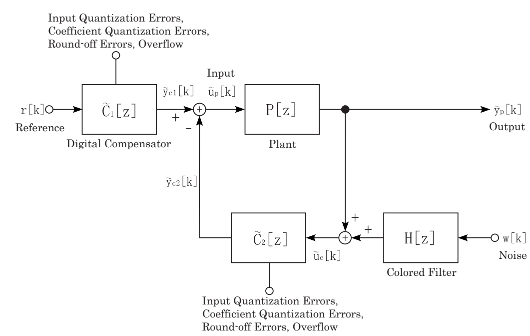

Fig. 1: Finite-Word-Length Control System

By using these representations, we have $\boldsymbol{x}_s = [\boldsymbol{x}_p^T[k]\boldsymbol{x}_{c1}^T[k]\boldsymbol{x}_{c2}^T[k]]^T$, $y_s[k] = y_p[k]$ and, further, we have the following augmented state equation, consisting of infinite-word-length (IWL) digital compensators and the plant:

$$S \left\{ \begin{array}{l} \boldsymbol{x}_s[k+1] = \boldsymbol{A}_s\boldsymbol{x}_s[k] + \boldsymbol{b}_s \left[\begin{array}{c} r[k] \\ w[k] \end{array} \right] \\ y_s[k] = \boldsymbol{c}_s\boldsymbol{x}_s[k] \end{array} \right. \tag{4}$$

where

$$\boldsymbol{A}_s = \left[\begin{array}{ccc} \boldsymbol{A}_p - d_{c2}\boldsymbol{b}_p\boldsymbol{c}_p & \boldsymbol{b}_p\boldsymbol{c}_{c1} & -\boldsymbol{b}_p\boldsymbol{c}_{c2} \\ \boldsymbol{0} & \boldsymbol{A}_{c1} & \boldsymbol{0} \\ \boldsymbol{b}_{c2}\boldsymbol{c}_p & \boldsymbol{0} & \boldsymbol{A}_{c2} \end{array} \right],$$

$$\boldsymbol{b}_s = \left[\begin{array}{cc} d_{c1}\boldsymbol{b}_p & -d_{c2}\boldsymbol{b}_p \\ \boldsymbol{b}_{c1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{b}_{c2} \end{array} \right], \quad \boldsymbol{c}_s = \left[\begin{array}{ccc} \boldsymbol{c}_p & 0 & 0 \end{array} \right]$$

Next, consider the FWL digital compensator. Let the digital compensators be $\tilde{C}_1$ and $\tilde{C}_2$, then, we have the following:

$$\tilde{C}_1 \begin{cases} \tilde{\boldsymbol{x}}_{c1}[k+1] = (\boldsymbol{A}_{c1} + \Delta\boldsymbol{A}_{c1}[k])\,\tilde{\boldsymbol{x}}_{c1}[k] \\ \qquad\qquad + (\boldsymbol{b}_{c1} + \Delta\boldsymbol{b}_{c1}[k])\, r[k] + \boldsymbol{\alpha}[k] \\ \tilde{y}_{c1}[k] = (\boldsymbol{c}_{c1} + \Delta\boldsymbol{c}_{c1}[k])\,\tilde{\boldsymbol{x}}_{c1}[k] \\ \qquad\qquad + (d_{c1} + \Delta d_{c1}[k])\, r[k] + \beta[k] \end{cases} \tag{5}$$

$$\tilde{C}_2 \begin{cases} \tilde{\boldsymbol{x}}_{c2}[k+1] = (\boldsymbol{A}_{c2} + \Delta\boldsymbol{A}_{c2}[k])\,\tilde{\boldsymbol{x}}_{c2}[k] \\ \qquad\qquad + (\boldsymbol{b}_{c2} + \Delta\boldsymbol{b}_{c2}[k])\, \tilde{u}_c[k] + \varepsilon[k] \\ \tilde{y}_{c2}[k] = (\boldsymbol{c}_{c1} + \Delta\boldsymbol{c}_{c2}[k])\,\tilde{\boldsymbol{x}}_{c2}[k] \\ \qquad\qquad + (d_{c2} + \Delta d_{c2}[k])\, \tilde{u}_c[k] + \eta[k] \end{cases} \tag{6}$$

where $\boldsymbol{\alpha}[k]$, $\beta[k]$, $\boldsymbol{\gamma}[k]$, and $\epsilon[k]$ are additive errors caused by round-off errors, multiplication and overflow of addition. $(\Delta\boldsymbol{A}_{c1}[k],\ \Delta\boldsymbol{b}_{c1}[k],\ \Delta\boldsymbol{c}_{c1}[k],\ \Delta d_{c1}[k])$, $(\Delta\boldsymbol{A}_{c2}[k],\ \Delta\boldsymbol{b}_{c2}[k],\ \Delta\boldsymbol{c}_{c2}[k],\ \Delta d_{c2}[k])$ are slightly changed in random, which are due to the coefficient quantization errors, are no relation each other, assume white noises, and are no relation with $w[k]$. And assume its average is 0 and its variance is ρ^2 $(0 < \rho \ll 1)$. Here, $\tilde{\boldsymbol{x}}_s = \left[\tilde{\boldsymbol{x}}_p^T[k]\tilde{\boldsymbol{x}}_{c1}^T[k]\tilde{\boldsymbol{x}}_{c2}^T[k]\right]^T$ and $\tilde{y}_s[k] = \tilde{y}_p[k]$ hold, we have the following augmented control system state equations consisting of the FWL digital controller and the plant:

$$\tilde{S} \begin{cases} \tilde{\boldsymbol{x}}_s[k+1] = (\boldsymbol{A}_s + \Delta\boldsymbol{A}_s[k])\,\tilde{\boldsymbol{x}}_s[k] \\ \qquad + (\boldsymbol{b}_s + \Delta\boldsymbol{b}_s[k]) \begin{bmatrix} r[k] & w[k] \end{bmatrix}^T \\ \qquad + \begin{bmatrix} \boldsymbol{b}_p^T\beta[k] - \boldsymbol{b}_p^T\eta[k] & \boldsymbol{\alpha}^T[k] & \varepsilon^T[k] \end{bmatrix} \\ \tilde{y}_s[k] = \boldsymbol{c}_s\tilde{\boldsymbol{x}}_s[k] \end{cases} \tag{7}$$

$$\Delta\boldsymbol{A}_s[k] = \begin{bmatrix} -\Delta d_{c2}\boldsymbol{b}_p\boldsymbol{c}_p & \boldsymbol{b}_p\Delta\boldsymbol{c}_{c1} & -\boldsymbol{b}_p\Delta\boldsymbol{c}_{c2} \\ \boldsymbol{0} & \Delta\boldsymbol{A}_{c1} & \boldsymbol{0} \\ \Delta\boldsymbol{b}_{c2}\boldsymbol{c}_p & \boldsymbol{0} & \Delta\boldsymbol{A}_{c2} \end{bmatrix},$$

$$\Delta\boldsymbol{b}_s[k] = \begin{bmatrix} \Delta d_{c1}\boldsymbol{b}_p & -\Delta d_{c2}\boldsymbol{b}_p \\ \Delta\boldsymbol{b}_{c1} & \boldsymbol{0} \\ \boldsymbol{0} & \Delta\boldsymbol{b}_{c2} \end{bmatrix}$$

3. ANALYSIS OF QUANTIZATION ERROR

The combination of (4) and (7) gives output error $\Delta y_s[k] = \widetilde{y}_s[k] - y_s[k]$

$$\begin{aligned} \boldsymbol{e}_{xs}[k+1] &= \tilde{\boldsymbol{x}}_s[k+1] - \boldsymbol{x}_s[k+1] \\ &= \boldsymbol{A}_s\boldsymbol{e}_{xs}[k] + \Delta\boldsymbol{A}_s[k]\boldsymbol{x}_s[k] + \Delta\boldsymbol{A}_s[k]\boldsymbol{e}_{xs}[k] \\ &+ \Delta\boldsymbol{b}_s \begin{bmatrix} \tilde{r}[k] \\ w[k] \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_p\beta[k] - \boldsymbol{b}_p\eta[k] \\ \boldsymbol{\alpha}[k] \\ \varepsilon[k] \end{bmatrix} \end{aligned} \tag{8}$$

$$\Delta y_s[k] = \boldsymbol{c}_s\tilde{\boldsymbol{e}}_{xs}[k] \tag{9}$$

In (8), since $\Delta\boldsymbol{A}_s[k]\boldsymbol{e}_{xs}[k]$ is smaller than other terms, we omit it from now on. Solving $\Delta y_s[k]$ in (8), (9) gives two terms due to the source of the errors:

$$\Delta y_s[k] = \Delta y_r[k] + \Delta y_k[k] \tag{10}$$

$$\Delta y_r[k] = \boldsymbol{c}_s \sum_{i=0}^{k-1} \boldsymbol{A}_s^{k-i-1} \begin{bmatrix} \boldsymbol{b}_p\beta[i] - \boldsymbol{b}_p\eta[i] \\ \boldsymbol{\alpha}[i] \\ \varepsilon[i] \end{bmatrix} \tag{11}$$

$$\begin{aligned} \Delta y_k[k] &= \boldsymbol{c}_s \sum_{i=0}^{k-1} \boldsymbol{A}_s^{k-i-1} \Big(\Delta\boldsymbol{A}_s[i]\boldsymbol{x}_s[i] \\ &\quad + \Delta\boldsymbol{b}_s[i] \begin{bmatrix} r[i] \\ w[i] \end{bmatrix} \Big) \end{aligned} \tag{12}$$

3.1 Analysis of the round-off errors

Assume the input signal $u[k]$ varies a lot in comparison with the quantization level. Then $\boldsymbol{\alpha}[k]$, $\beta[k]$, $\varepsilon[k]$, and $\eta[k]$ become random noises and $\Delta y_r[k]$ is called round-off error. Consequently, when such an input signal is filtered, the sources of round-off error of $(\boldsymbol{A}_s,\ \boldsymbol{b}_s,\ \boldsymbol{c}_s,\ d)$, and each element of $\boldsymbol{\alpha}[k]$, $\beta[k]$, $\varepsilon[k]$, and $\eta[k]$ can be assumed distributed like white noises which varies within $[-2^{-l}/2, 2^{-l}/2]$ and are independent of each other. Thus, when $\sigma^2 = 2^{-l}/12$, the following equation can be denoted:

$$\mathrm{E}\left[\begin{pmatrix} \boldsymbol{\alpha}[i] \\ \beta[i] \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}[j] \\ \beta[j] \end{pmatrix}^T \right] = \sigma^2\delta(i-j) \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & q \end{bmatrix} \tag{13}$$

$$\mathrm{E}\left[\begin{pmatrix} \varepsilon[i] \\ \eta[i] \end{pmatrix} \begin{pmatrix} \varepsilon[j] \\ \eta[j] \end{pmatrix}^T \right] = \sigma^2\delta(i-j) \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{0} \\ \boldsymbol{0} & z \end{bmatrix} \tag{14}$$

where $\boldsymbol{Q},\ \boldsymbol{Z}$ are diagonal matrix whose ith diagonal elements are given by the number of noninteger of the ith row of the matrix $\boldsymbol{A}_{c1},\ \boldsymbol{b}_{c1}$ and $\boldsymbol{A}_{c2},\ \boldsymbol{b}_{c2}$, respectively. Furthermore, q and z are the number of the noninteger in $\boldsymbol{c}_{c1},\ d_{c1}$ and $\boldsymbol{c}_{c2}$, We solve the variance $\mathrm{E}\,[\Delta y_r^2[k]]$ of $\Delta y_r[k]$ as

$$\begin{aligned} \mathrm{E}\Big[\Delta y_r^2[k]\Big] &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \boldsymbol{c}_s\boldsymbol{A}_s^{k-i-1} \\ &\times \mathrm{E}\left[\begin{bmatrix} \boldsymbol{b}_p\boldsymbol{b}_p^T\beta[i]\beta[j] + \boldsymbol{b}_p\boldsymbol{b}_p^T\eta[i]\eta[j] & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\alpha}[i]\boldsymbol{\alpha}[j] & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \varepsilon[i]\varepsilon[j] \end{bmatrix} \right] \\ &\times \left(\boldsymbol{A}_s^{k-j-i}\right)^T \boldsymbol{c}_s^T \end{aligned} \tag{15}$$

The equations (13) and (14) gives the variance of the steady-state round-off error $\Delta y_r^2[k]$ as follows:

$$\begin{aligned} \mathrm{E}\Big[\Delta y_r^2\Big] &= \sigma^2 \sum_{i=0}^{\infty} \boldsymbol{c}_s\boldsymbol{A}_s^i \\ &\times \begin{bmatrix} \boldsymbol{b}_p\boldsymbol{b}_p^Tq + \boldsymbol{b}_p\boldsymbol{b}_p^Tz & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z} \end{bmatrix} \left(\boldsymbol{A}_s^i\right)^T \boldsymbol{c}_s^T \end{aligned} \tag{16}$$

Since (16) is scalar and its trace is the same,

$$\mathrm{E}\Big[\Delta y_r^2\Big] = \sigma^2 \mathrm{tr}\left[\begin{bmatrix} \boldsymbol{b}_p\boldsymbol{b}_p^Tq + \boldsymbol{b}_p\boldsymbol{b}_p^Tz & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z} \end{bmatrix} \boldsymbol{W} \right] \tag{17}$$

where $\boldsymbol{W} = \sum_{i=0}^{\infty} \left(\boldsymbol{A}_s^i\right)^T \boldsymbol{c}_s^T\boldsymbol{c}_s\boldsymbol{A}_s^i$ $\boldsymbol{W}$ is called the noise matrix, which can be obtained as a solution of the following Lyapunov equation:

$$\boldsymbol{W} = \boldsymbol{A}_s^T\boldsymbol{W}\boldsymbol{A}_s + \boldsymbol{c}_s^T\boldsymbol{c}_s \tag{18}$$

In (17), σ^2 is obtained by the word length l, which does not change even if the structure of the compensator is changed. Thus dividing (17) by σ^2 gives

$$G_s = \mathrm{tr}\left[\begin{bmatrix} \boldsymbol{b}_p\boldsymbol{b}_p^Tq + \boldsymbol{b}_p\boldsymbol{b}_p^Tz & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z} \end{bmatrix} \boldsymbol{W} \right] \tag{19}$$

where G_s is called the noise power gain.

3.2 Analysis of the coefficient quantization error

In (8) assume $\boldsymbol{\alpha}[k]=0,\ \beta[k]=0,\ \varepsilon[k]=0,\ \eta[k]=0$ and we analyze the output error due to quantization of coefficiency. Let $\boldsymbol{V}_{exs}[k]=\mathrm{E}\left[e_{xs}[k]e_{xs}^T\right]$ and

$$\begin{aligned}\mathrm{E}\Big[\Delta y_k^2[k]\Big] &= \mathrm{E}\Big[\{\boldsymbol{c}_s\boldsymbol{e}_{xs}[k]\}\{\boldsymbol{c}_s\boldsymbol{e}_{xs}[k]\}^T\Big]\\ &= \boldsymbol{c}_s\boldsymbol{V}_{exs}[k]\boldsymbol{c}_s^T \qquad (20)\end{aligned}$$

The equation (8) gives,

$$\begin{aligned}\boldsymbol{V}_{exs}[k+1] &= \mathrm{E}\Big[\boldsymbol{e}_{xs}[k+1]\boldsymbol{e}_{xs}^T[k+1]\Big]\\ &= \boldsymbol{A}_s\mathrm{E}\Big[\boldsymbol{e}_{xs}[k]\boldsymbol{e}_{xs}^T[k]\Big]\boldsymbol{A}_s^T\\ &\quad +\mathrm{E}\Big[\Delta\boldsymbol{A}_s\boldsymbol{e}_{xs}[k]\boldsymbol{e}_{xs}^T[k]\Delta\boldsymbol{A}_s^T\Big]\\ &\quad +\mathrm{E}\Big[\Delta\boldsymbol{b}_s\Delta\boldsymbol{b}_s^T w^2[k]\Big]\\ &\quad +\mathrm{E}\Big[\Delta\boldsymbol{b}_s\Delta\boldsymbol{b}_s^T r^2[k]\Big]\\ &= \boldsymbol{A}_s\boldsymbol{V}_{exs}[k]\boldsymbol{A}_s^T+\boldsymbol{L} \qquad (21)\end{aligned}$$

Here, assume $r[k]=0$ then $\mathrm{E}\left[\Delta\boldsymbol{b}_s\Delta\boldsymbol{b}_s^T r^2[k]\right]=0$ and

$$\begin{aligned}\boldsymbol{L} &= \mathrm{E}\left[\Delta\boldsymbol{A}_s\boldsymbol{e}_{xs}[k]\boldsymbol{e}_{xs}^T[k]\Delta\boldsymbol{A}_s^T\right]\\ &\quad +\mathrm{E}\left[\Delta\boldsymbol{b}_s\Delta\boldsymbol{b}_s^T w^2[k]\right]\\ &= \sigma^2\begin{bmatrix}\boldsymbol{L}_{11} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{L}_{22} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_{33}\end{bmatrix} \qquad (22)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{L}_{11} &= \gamma(d_{c2})\,\boldsymbol{b}_p\boldsymbol{b}_p^T\boldsymbol{c}_p\tilde{\boldsymbol{K}}_p\boldsymbol{c}_p^T+\boldsymbol{b}_p\boldsymbol{b}_p^T\sum_{i=1}^{n}\gamma(c_{c1i})\,\tilde{\boldsymbol{K}}_{c1ii}\\ &\quad +\boldsymbol{b}_p\boldsymbol{b}_p^T\sum_{i=1}^{n}\gamma(c_{c2i})\,\tilde{\boldsymbol{K}}_{c2ii}+\sigma_w^2\gamma(d_{c1})\,\boldsymbol{b}_p\boldsymbol{b}_p^T\\ &\quad +\sigma_w^2\gamma(d_{c2})\,\boldsymbol{b}_p\boldsymbol{b}_p^T \qquad (23)\end{aligned}$$

$$\begin{aligned}\boldsymbol{L}_{22} &= \mathrm{diag}\left(\sum_{i=1}^{n}\gamma(A_{c11i})\,\tilde{\boldsymbol{K}}_{c1ii},\cdots,\right.\\ &\qquad\qquad \left.\sum_{i=1}^{n}\gamma(A_{c1ni})\,\tilde{\boldsymbol{K}}_{c1ii}\right)\\ &\quad +\sigma_w^2\mathrm{diag}\Big(\gamma(b_{c11}),\cdots,(b_{c1n})\Big) \qquad (24)\end{aligned}$$

$$\begin{aligned}\boldsymbol{L}_{33} &= \mathrm{diag}\left(\gamma(b_{c21})\,\boldsymbol{c}_p\tilde{\boldsymbol{K}}_p\boldsymbol{c}_p^T+\sum_{i=1}^{n}\gamma(A_{c21i})\,\tilde{\boldsymbol{K}}_{c2ii},\right.\\ &\qquad \left.\cdots,\gamma(b_{c2n})\,\boldsymbol{c}_p\tilde{\boldsymbol{K}}_p\boldsymbol{c}_p^T+\sum_{i=1}^{n}\gamma(A_{c2ni})\,\tilde{\boldsymbol{K}}_{c2ii}\right)\\ &\quad +\sigma_w^2\mathrm{diag}\Big(\gamma(b_{c21}),\cdots,(b_{c2n})\Big) \qquad (25)\end{aligned}$$

$\gamma(*)$ in (23), (24), and (25) is a function representing the source of the error and it is given by

$$\gamma(*)=\begin{cases}0 & *=\text{Integer}\\ 1 & *\neq\text{Integer}\end{cases} \qquad (26)$$

where $\tilde{\boldsymbol{K}}_p$ is the covariance matrix of the state of the plant $\boldsymbol{x}_p[k]$ against colored noise. $\tilde{\boldsymbol{K}}_{c1}$ and $\tilde{\boldsymbol{K}}_{c2}$ are the covariance matrix of the digital compensators $\tilde{C}_1$ and $\tilde{C}_2$ is $\tilde{\boldsymbol{K}}_{c1}=\mathrm{E}\left[\boldsymbol{x}_{c1}[k]\boldsymbol{x}_{c1}^T[k]\right]$ and $\tilde{\boldsymbol{K}}_{c2}=\mathrm{E}\left[\boldsymbol{x}_{c2}[k]\boldsymbol{x}_{c2}^T[k]\right]$, respectively. $\tilde{\boldsymbol{K}}_{c1ii}$ and $\tilde{\boldsymbol{K}}_{c2ii}$ are i th diagonal element of $\tilde{\boldsymbol{K}}_{c1}$ and $\tilde{\boldsymbol{K}}_{c2}$. Furthermore, σ_w^2 is the variance of the output of the coloring filter $w[k]$.
If $\boldsymbol{A}_s$ is stable, the covariance matrix $\boldsymbol{V}_{exs}[k]$ converges. Thus, when the value of the steady state is $\boldsymbol{V}_{exs}$, it can be obtained by solving the following Lyapunov equation:

$$\boldsymbol{V}_{exs}=\boldsymbol{A}_s\boldsymbol{V}_{exs}\boldsymbol{A}_s^T+\sigma^2\boldsymbol{L} \qquad (27)$$

Thus, $\mathrm{E}\left[\Delta y_k^2\right]$can be obtained by

$$\begin{aligned}\mathrm{E}\left[\Delta y_k^2\right] &= \boldsymbol{c}_s\boldsymbol{V}_{exs}\boldsymbol{c}_s^T\\ &= \sigma^2 tr\left[\boldsymbol{L}\sum_{i=0}^{\infty}\left(\boldsymbol{A}_s^i\right)^T\boldsymbol{c}_s^T\boldsymbol{c}_s\boldsymbol{A}_s^i\right]\\ &= \sigma^2 tr\left[\boldsymbol{L}\boldsymbol{W}\right] \qquad (28)\end{aligned}$$

where $\boldsymbol{W}=\boldsymbol{A}_s^T\boldsymbol{W}\boldsymbol{A}_s+\boldsymbol{c}_s^T\boldsymbol{c}_s$ is the noise matrix (observability Grammian) . Since σ^2 is determined by the word length ℓ, it does not change even if the structure of the digital compensator is changed. Thus divide (28) by σ^2, we call it the stastistical coefficient sensitivity

$$S_k=tr\left[\boldsymbol{L}\boldsymbol{W}\right] \qquad (29)$$

4. THE PROPOSAL OF OPTIMAL STRUCTURE

4.1 The state transformation of the control system

In this section we propose the structure of the control system. Note that we equivalently transform only the compensators. We define the state transformation matrix as

$$\boldsymbol{T}=\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{T}_{c1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_{c2}\end{bmatrix} \qquad (30)$$

where $\boldsymbol{I}\in R^{m\times m}$, $\boldsymbol{T}_{c1}\in R^{n_1\times n_1}$, and $\boldsymbol{T}_{c2}\in R^{n_2\times n_2}$

4.2 Valuation of noise power gain and coefficient sensitivity

First, we obtain the noise power gain G_s' of the equivalently transformed control system. Since we can have noise power matrix $\boldsymbol{W'}$ is $\boldsymbol{W'}=\boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T}$easily,

$$\begin{aligned}G_s' &= \mathrm{tr}\left[\begin{bmatrix}\boldsymbol{b}_p\boldsymbol{b}_p^Tq'+\boldsymbol{b}_p\boldsymbol{b}_p^Tz' & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T}\right]\\ &\quad +\mathrm{tr}\left[\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q'} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Z'}\end{bmatrix}\boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T}\right] \qquad (31)\end{aligned}$$

where $\boldsymbol{Q'}$, q', $\boldsymbol{Z'}$, and z' are the matrix representing the number of the sources of round off. Since the equation (31) reveals that the noise power gain G_s' contains a state transformation matrix, the round off errors depend on the compensators.

Note the first term of the right hand of (31). To show it doesn't change after the state transformation $\boldsymbol{T}$, we partition $\boldsymbol{W}$ as

$$\boldsymbol{W}=\begin{bmatrix}\boldsymbol{W}_{11} & \boldsymbol{W}_{12} & \boldsymbol{W}_{13}\\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} & \boldsymbol{W}_{23}\\ \boldsymbol{W}_{31} & \boldsymbol{W}_{32} & \boldsymbol{W}_{33}\end{bmatrix} \tag{32}$$

where $\boldsymbol{W}_{11}\in R^{m\times m}$, $\boldsymbol{W}_{12}\in R^{m\times n_1}$, $\boldsymbol{W}_{13}\in R^{m\times n_2}$, $\boldsymbol{W}_{21}\in R^{n_1\times m}$, $\boldsymbol{W}_{22}\in R^{n_1\times n_1}$, $\boldsymbol{W}_{23}\in R^{n_1\times n_2}$, $\boldsymbol{W}_{31}\in R^{n_2\times m}$, $\boldsymbol{W}_{32}\in R^{n_2\times n_1}$, and $\boldsymbol{W}_{33}\in R^{n_2\times n_2}$, respectively.

Rewrite the first term of the right hand of (31) by these partitionized noise matrix $\boldsymbol{W}$. Since we calculate the trace, state transformation matrix$\boldsymbol{T}_{c1}$ is not contained in the term. Therefore, the state transformation doesn't affect the noise power gain G'_s and we note only the second term. Thus, instead of minimization of G'_s, minimize

$$G'_{ss}=tr\left[\boldsymbol{Q'}\boldsymbol{T}_{c1}^T\boldsymbol{W}_{22}\boldsymbol{T}_{c1}\right] \tag{33}$$

Second, we evaluate the coefficient sensitivity:

$$\begin{aligned}S'_k &= \begin{bmatrix}\boldsymbol{L}_{11} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T}\\ &\quad+\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{L}_{22} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}_{33}\end{bmatrix}\boldsymbol{T}^T\boldsymbol{W}\boldsymbol{T}\end{aligned} \tag{34}$$

In this case, also, the first term of the right hand of (34) doesn't change owing to the state transformation matrix $\boldsymbol{T}$. Thus, we note the second term in the case of derivation of optimal structure instead of minimization of S'_k. Consider to minimize

$$S'_{kss}=\mathrm{tr}\left[\boldsymbol{L'}_{22}\boldsymbol{T}_{c1}^T\boldsymbol{W}_{22}\boldsymbol{T}_{c1}\right] \tag{35}$$

4.3 Scaling

Third, we consider scaling to prevent overflow. Assume the FWL state variable $\boldsymbol{x}_s[k]$ is within $[-1,+1]$.The variance $\mathrm{E}\left[x_{si}^2\right]$ of the state variable $x_{si}\,(i=1,2\cdots,n)$ of $\boldsymbol{A}_s$, $\boldsymbol{b}_s$, $\boldsymbol{c}_s$、d_s) is given by the diagonal elements of the covariance matrix $\mathrm{E}\left[\boldsymbol{x}_s[k]\boldsymbol{x}_s^T[k]\right]$. The state variable that can be denoted is given by

$$\boldsymbol{x}_s[k]=\sum_{i=0}^{k-1}\boldsymbol{A}_s^{k-i-1}\boldsymbol{b}_s w[k] \tag{36}$$

The variance $\mathrm{E}\left[\boldsymbol{x}_s[k]\boldsymbol{x}_s^T[k]\right]$ of $\boldsymbol{x}_s[k]$ is given as

$$\begin{aligned}&\mathrm{E}\Big[\boldsymbol{x}_s[k]\boldsymbol{x}_s^T[k]\Big]\\ &=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\boldsymbol{A}_s^{k-i-1}\boldsymbol{b}_s\mathrm{E}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & w[i]w[j]\end{bmatrix}\\ &\times\boldsymbol{b}_s^T\left(\boldsymbol{A}_s^{k-j-1}\right)^T\end{aligned} \tag{37}$$

Here, combine as the differences between i and j are constants, $\boldsymbol{K}_\rho$, which is $\rho=i-j>0$, is denoted

$$\begin{aligned}\boldsymbol{K}_\rho[k] &= \sum_{i=\rho}^{k-1}\boldsymbol{A}_s^{k-j-1}\boldsymbol{b}_s\mathrm{E}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & w[i]w[i-\rho]\end{bmatrix}\\ &\quad\times\boldsymbol{b}_s^T\left(\boldsymbol{A}_s^{k-i-1}\right)^T\left(\boldsymbol{A}_s^\rho\right)^T\end{aligned} \tag{38}$$

In case of steady state, i.e. $k\to\infty$

$$\lim_{k\to\infty}\sum_{i=\rho}^{k-1}\boldsymbol{A}^{k-i-1}=\sum_{i=0}^{\infty}\boldsymbol{A}^i \tag{39}$$

holds, (38) in steady state can be expressed as

$$\begin{aligned}&\lim_{k\to\infty}\boldsymbol{K}_\rho=\\ &\sum_{i=0}^{\infty}\boldsymbol{A}_s^i\boldsymbol{b}_s\mathrm{E}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & w[i]w[i-\rho]\end{bmatrix}\boldsymbol{b}_s^T\left(\boldsymbol{A}_s^i\right)^T\left(\boldsymbol{A}_s^\rho\right)^T\end{aligned} \tag{40}$$

Assume the eigen values of $\boldsymbol{A}_H$ differ for each other and let diagonalizing matrix of $\boldsymbol{A}_H$ be $\boldsymbol{V}$ and the corresponding eigen values be, $m_1,\cdots,m_{nH}$, then

$$\begin{aligned}\mathrm{E}\Big[w[k]w[k-\rho]\Big] &= \boldsymbol{c}_H\boldsymbol{A}_H^\rho\boldsymbol{K}_H\boldsymbol{c}_H^T+\boldsymbol{c}_H\boldsymbol{A}_H^{\rho-1}\boldsymbol{b}_H d_H\\ &= \sum_{i=1}^{n_H}m_i^\rho Z_i\end{aligned} \tag{41}$$

where Z_i is the ith diagonal element of $\boldsymbol{Z}=\boldsymbol{V}^{-1}\left(\boldsymbol{K}_H\boldsymbol{c}_H^T+\boldsymbol{A}_H^{-1}\boldsymbol{b}_H d_H\right)\boldsymbol{c}_H\boldsymbol{V}$. Then substitution (41) into (40) gives

$$\lim_{k\to\infty}\boldsymbol{K}_\rho=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sum_{i=1}^{n_H}m_i^\rho Z_i\end{bmatrix}\boldsymbol{K}_s\left(\boldsymbol{A}_{si}^\rho\right)^T \tag{42}$$

$\boldsymbol{A}_{si}$ is given by $m_i\boldsymbol{A}_s$ and $\boldsymbol{K}_s$ is given by $\sum_{i=0}^{\infty}\boldsymbol{A}_s^i\boldsymbol{b}_s\boldsymbol{b}_s^T\left(\boldsymbol{A}_s^i\right)^T$, which is the variance of the state variable of the control system against white noise. Since the terms which satisfy $\rho<0$ are symmetric matrix of $\boldsymbol{K}_\rho\,(\rho>0)$,

$$\boldsymbol{K}_\rho=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sum_{i=1}^{n_H}m_i^\rho Z_i\end{bmatrix}\left(\boldsymbol{A}_{si}^\rho\right)^T\boldsymbol{K}_s \tag{43}$$

holds. On the other hand, the term $\boldsymbol{K}_0$, i.e. when $\rho=0$, can be obtained in steady state as follows:

$$\boldsymbol{K}_0=\sum_{i=0}^{\infty}\boldsymbol{A}_s^i\boldsymbol{b}_s\mathrm{E}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & w[i]w[i]\end{bmatrix}\boldsymbol{b}_s^T\left(\boldsymbol{A}_s^i\right)^T \tag{44}$$

where

$$\mathrm{E}\Big[w^2[k]\Big] = \sigma_w^2=\boldsymbol{c}_H\boldsymbol{K}_H\boldsymbol{c}_H^T+d_Hd_H \tag{45}$$

$$\boldsymbol{K}_0 = \sigma_w^2\boldsymbol{K}_s \tag{46}$$

Here, σ_w^2 is the variance of the colored noise. Thus, the variance of the state variable of the control system is given by

$$\begin{aligned}\tilde{\boldsymbol{K}}_s &= \mathrm{E}\Big[\boldsymbol{x}_s[k]\boldsymbol{x}_s^T[k]\Big]\\ &= \sum_{i=1}^{n_H}Z_i\boldsymbol{K}_s\left(\boldsymbol{A}_{si}^\infty\right)+\cdots+\sum_{i=1}^{n_H}Z_i\boldsymbol{K}_s\boldsymbol{A}_{si}^T\\ &\quad+\sum_{i=1}^{n_H}Z_i\boldsymbol{A}_{si}\boldsymbol{K}_s+\cdots+\sum_{i=1}^{n_H}Z_i\boldsymbol{A}_{si}^\infty\boldsymbol{K}_s+\sigma_w^2\boldsymbol{K}_s\\ &= \sum_{i=1}^{n_H}Z_i\left\{\boldsymbol{K}_s\boldsymbol{A}_{si}^T\left(\boldsymbol{I}-\boldsymbol{A}_{si}^T\right)^{-1}\right.\\ &\quad\left.+\left(\boldsymbol{I}-\boldsymbol{A}_{si}\right)^{-1}\boldsymbol{A}_{si}\boldsymbol{K}_s\right\}+\sigma_w^2\boldsymbol{K}_s\end{aligned} \tag{47}$$

Next, the variance of the state variable $\mathrm{E}\left[x'^2_{si}\right]$ after state transformation is given by

$$\tilde{\boldsymbol{K}}'_s = \boldsymbol{T}^{-1}\tilde{\boldsymbol{K}}_s\boldsymbol{T}^{-T} \tag{48}$$

4.4 The optimal structure

Compare (33) and (35) then G'_{ss} has the same form as S'_{kss} if $\boldsymbol{Q}'$ and $\boldsymbol{L}'_{22}$ are neglected. Actually, $\boldsymbol{Q}'$ represents the number of round- off errors, on the other hand, $\boldsymbol{L}'_{22}$ represents the number of coefficient errors. Such noninteger coefficient causes round off errors, and on the other hand, the noninteger coefficients are affected by quantization. Thus, the numbers of the sources of round-off errors and the sources of coefficient errors are the same. Consequently, it is reasonable that we consider either round-off errors or coefficient quantization errors.

The optimal structure problem is formulated as follows:
"Solve the state transformation matrix $\boldsymbol{T}_{c1}$ which minimizes

$$\tilde{G}'_{ss} = \mathrm{tr}\left[\boldsymbol{Q}'\boldsymbol{T}_{c1}^T\tilde{\boldsymbol{W}}_{22}\boldsymbol{T}_{c1}\right] \tag{49}$$

such that

$$\boldsymbol{T}_{c1}^{-1}\tilde{\boldsymbol{K}}_c\boldsymbol{T}_{c1}^{-T} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix}\text{"} \tag{50}$$

Except some special cases, $\boldsymbol{Q}'_{ii} = n+1, q' = n+1, \boldsymbol{Z}'_{ii} = \boldsymbol{n+1}, \boldsymbol{z}' = \boldsymbol{n+1}$ hold, so we can change the problems as follows:
"Minimize

$$\tilde{G}_0 = \mathrm{tr}\left[\boldsymbol{T}_{c1}^T\tilde{\boldsymbol{W}}_{22}\boldsymbol{T}_{c1}\right] \tag{51}$$

such that

$$\boldsymbol{T}_{c1}^{-1}\tilde{\boldsymbol{K}}_c\boldsymbol{T}_{c1}{}^{-T} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix}\text{"} \tag{52}$$

First, let state transformation matrix be $\boldsymbol{T}_{c1} = \boldsymbol{T}_0\boldsymbol{T}_1$ where $\boldsymbol{T}_0$ satisfies

$$\boldsymbol{K}_0 = \boldsymbol{T}_0^{-1}\tilde{\boldsymbol{K}}_c\left(\boldsymbol{T}_0^{-1}\right)^T = \boldsymbol{I} \tag{53}$$

$\boldsymbol{T}_0$ converts $\boldsymbol{W}_{22}$ as

$$\boldsymbol{W}_0 = \boldsymbol{T}_0^T\boldsymbol{W}_{22}\boldsymbol{T}_0 \tag{54}$$

Let $\boldsymbol{T}_1$ be an arbitrary real invertible matrix and, by using an orthogonal matrix $\boldsymbol{R}_0$ and positive matrix $\boldsymbol{S}$, be

$$\boldsymbol{T}_1 = \boldsymbol{R}_0\boldsymbol{S} \tag{55}$$

Since $\boldsymbol{S}$ is positive, it can be analyzed by an orthogonal matrix $\boldsymbol{R}$ and a diagonal matrix $\boldsymbol{\Lambda}$ then letting $\boldsymbol{R}_1 = \boldsymbol{R}\boldsymbol{R}_0$ gives

$$\boldsymbol{T}_1 = \boldsymbol{R}_1\boldsymbol{\Lambda}\boldsymbol{R}_0^T \tag{56}$$

Further, substituting (53) and (56) into $G'_{ss} = \mathrm{tr}[\boldsymbol{T}_{c1}\boldsymbol{T}_{c1}]$ gives

$$\boldsymbol{R}_0\boldsymbol{\Lambda}^{-2}\boldsymbol{R}_0^T = \begin{bmatrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{bmatrix} \tag{57}$$

$$G'_0 = \sum_{i=1}^{n}\tilde{\lambda}_i^2\tilde{\gamma}_i^2 \tag{58}$$

$\tilde{\lambda}_i^2$ and $\tilde{\gamma}_i^2$ are the ith diagonal elements of $\boldsymbol{\Lambda}^2$ and $\boldsymbol{R}_1^T\boldsymbol{W}_0\boldsymbol{R}_1$,respectively. The trace and determinant of this equation and this inequality leads to

$$\sum_{i=1}^{n}\frac{1}{\tilde{\lambda}_i^2} = n, \quad \prod_{i=1}^{n}\tilde{\lambda}_i^2 \geq n \tag{59}$$

By using this equation and this inequality, solving minimization of $\tilde{G}_0$ can be changed as follows:
"obtain $\boldsymbol{T}_c$ which minimizes

$$\text{"}\tilde{G}_0 = \sum_{i=1}^{n}\tilde{\lambda}_i^2\tilde{\gamma}_i^2 \tag{60}$$

such that

$$\sum_{i=1}^{n}\frac{1}{\tilde{\lambda}_i^2} = n \tag{61}$$

Solving this problem by using Ragrange's method gives when

$$\tilde{\lambda}_i = \left(\sum_{j=1}^{n}\tilde{\gamma}_j/n\tilde{\gamma}_i\right)^{1/2}$$

and minimized value is

$$\left(\sum_{i=1}^{n}\tilde{\gamma}_i\right)^2/n$$

Next, we consider the minimum value $\tilde{G}_0$ with respect to $\tilde{\gamma}$. In order to do this, consider an orthogonal matrix $\boldsymbol{S}$ diogonalizing $\boldsymbol{R}_1^T\boldsymbol{W}_0\boldsymbol{R}_1$. Since $\boldsymbol{R}_1^T\boldsymbol{W}_0\boldsymbol{R}_1$is a symmetric matrix, there exist inevitably orthogonal matrix$\boldsymbol{S}$. Let the eigen values of $\boldsymbol{W}_0$ be $\tilde{\theta}_1^2, \cdots, \tilde{\theta}_n^2$. Calculating the trace and determinant of $\boldsymbol{R}_1^T\boldsymbol{W}_0\boldsymbol{R}_1$gives

$$\sum_{i=1}^{n}\tilde{\gamma}_i^2 = \sum_{i=1}^{n}\tilde{\theta}_i^2 = \mathrm{tr}\left(\boldsymbol{W}_0\right) = \text{constant} \tag{62}$$

$$\prod_{i=1}^{n}\tilde{\gamma}_i^2 \geq \prod_{i=1}^{n}\tilde{\theta}_i^2 = \det\left(\boldsymbol{W}_0\right) = \text{constant} \tag{63}$$

This equation lead to

$$\sum_{i=1}^{n}\tilde{\gamma}_i \geq \sum_{i=1}^{n}\tilde{\theta}_i \tag{64}$$

Thus, when $\tilde{\gamma}_i = \tilde{\theta}_i$,

$$\tilde{G}_0 = \left(\sum_{i=1}^{n}\tilde{\theta}_i\right)^2/n \tag{65}$$

holds and $\tilde{G}_0$ is minimized. So, the noise power gain of the optimal structure by using the state transformation matrix $\boldsymbol{T}_{c1}$ is given by

$$\tilde{G}_{ss} = (n+1)\frac{\left(\sum_{i=1}^{n}\tilde{\theta}_i\right)^2}{n} \tag{66}$$

where $\tilde{\theta}_i$ is the root of the eigenvalue of $\tilde{\boldsymbol{K}}_c\boldsymbol{W}_{22}$ which is called the second mode of the compensators, and it does not depend on the structure of the compensators, and it is determined by the transfer function of the compensators. This fact is clarified by $\tilde{\boldsymbol{K}}'_c \boldsymbol{W}'_{22} = \boldsymbol{T}_{c1}^{-1} \tilde{\boldsymbol{K}}_c \boldsymbol{W}_{22} \boldsymbol{T}_{c1}$. The state transformation matrix $\boldsymbol{T}_{c1}$ can expressed by

$$\boldsymbol{T}_{c1} = \boldsymbol{T}_0 \boldsymbol{R}_1 \boldsymbol{\Lambda} \boldsymbol{R}_0^T \tag{67}$$

where

$\boldsymbol{T}_0$: $\left(\tilde{\boldsymbol{K}}_c\right)^{1/2}$

$\boldsymbol{R}_1$: the orthogonal matrix which consists of the eigen vector of $\boldsymbol{W}_0$

$\boldsymbol{\Lambda}$: whose ith diagonal element is given by

$$\tilde{\lambda}_i \left(\sum_{j=1}^{n} \tilde{\theta}_j / n \tilde{\theta}_i \right)^{1/2} \tag{68}$$

$\boldsymbol{R}_0$: given by $n-1$ rotation matrix and one of them is given by$\boldsymbol{R}_i$

$$\boldsymbol{R}_0 = \boldsymbol{R}_n \boldsymbol{R}_{n-1} \cdots \boldsymbol{R}_2 \tag{69}$$

$$\boldsymbol{R}_i = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{jj} & \mathbf{0} & R_{jk} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R_{kj} & \mathbf{0} & R_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{bmatrix} \tag{70}$$

$$R_{jj} = R_{kk} = \left(\frac{\mu_j^2 - 1}{\mu_j^2 - \mu_k^2} \right)^{1/2} R_{jk} = -R_{kj} = \left(\frac{1 - \mu_k^2}{\mu_j^2 - \mu_k^2} \right)^{1/2}$$

μ_j^2 and μ_k^2 are jth and kth diagonal elements $\boldsymbol{R}_{i-1} \cdots \boldsymbol{R}_2 \boldsymbol{\Lambda}^{-2} \boldsymbol{R}_2^T \cdots \boldsymbol{R}_{i-1}^T$, respectively where $\mu_j > 1$, $\mu_k < 1$.

5. SIMULATION RESULT

In this section, the four structures is verified, i.e. the directII structure, the pararell structure, the balanced realization structure, and the proposed optimal structure, $\tilde{C}_2$ is chosen by using H_∞ techniques, $\tilde{C}_1$ is chosen by model-matching scheme.

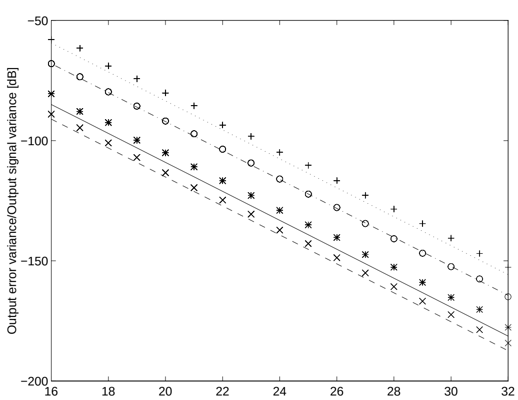

Fig. 2: Round-Off Errors

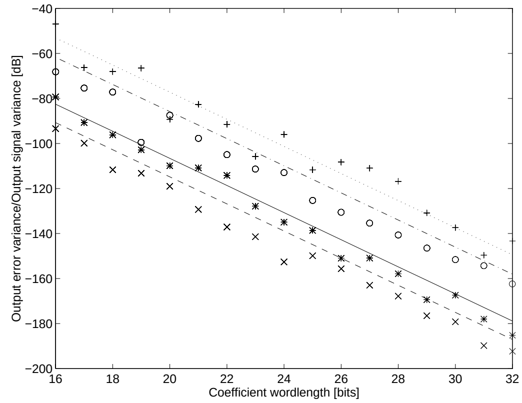

Fig. 3: Coefficient Quantization Errors

The direct II structure :
the ideal value $-$ the dotted line
simulated value $-+$
The parallel structure :
the ideal value $-$ the dash-and-dotted-line
simulated value $-$ o
the balanced realization structure :
ideal value $-$ the solid line, simulated value $- *$
optimal structure :
ideal value $-$ the broken line, simulated value $-$ x

6. CONCLUSION

In this paper, we dealt with the fixed-point two-degree-of-freedom control system and proposed the optimal structure which minimizes round-off error and coeffiecient-quantization error.
Simulation results indicate that the ideal value and simulated value are roughly identical and the proposed method is optimal within 4 structures.

REFERENCES

[1] D. Williamson, “Digital Control and Implementation”, Prenticehall (1991).

[2] M. Kawamata and T. Higuchi, “Quantization Effects in Digital Control”, J. of the Society of Instrument and Control Engineers, **27**, 12, pp.1091-1097 (1988).

[3] K. Ohkubo, “An Optimal Structure of Digital Filters Against Colored Noises”, Metrology Society of Australia Biennial International Measurement Conference in Sydney, pp.281-285 (1999).

[4] S. Y. Hwang, “Minimum Uncorrelated Unit Noise in State-Space Digital Filtering ”, IEEE Transaction on Acoustic, Speech, and Signal Processing, **ASSP-25**, 4, pp.273-281 (1977).