On the stabilization of singular bilinear systems

Jia-Rong Liang*, Ho-Lim Choi**, and Jong-Tae LIM**

*College of Computer and Information Engineering, GuangXi University, Nanning, China

**Department of Electrical Engineering, KAIST, Daejeon 305-701, Korea

(Tel: +82-42-869-3441; Fax: +82-42-869-3410; Email: jtlim@stcon.kaist.ac.kr)

Abstract: In this paper, the stability problem for singular bilinear system is investigated. We present state feedback control laws for two classes of singular bilinear plants. Asymptotic stability of the closed-loop systems is derived by employing singular Lyapunov's direct method. The primary advantage of our approach lies in its simplicity. In order to verify effectiveness of the results, two numerical examples are given.

Keywords: singular bilinear systems, stability, singular Lyapunov's function, state feedback.

1. Introduction

In the past decades, the stability problem of bilinear systems has been a topic of recurring interest. the bilinear systems may be modeled in engineering, biology, ecology, socioeconomic, nuclear, thermal and chemical process [1]. There are many results on the stability of bilinear systems, for example, [1] investigates the robust stability problems of a class of singularly perturbed discrete bilinear systems; [2] exhibits a bang-bang structure using a linear switching function to design stable feedback control of bilinear systems; [3] gives necessary and sufficient conditions for the asymptotic stabilization by using homogenous feedbacks; [4] presented a solution to the global stabilization problem via smooth state feedback by using KYP lemma. It should be pointed out that these results were concerned with regular bilinear systems. It is known that singular systems better describe physical systems than regular ones [5]. So far, there has been much literature to extend successfully the results on the stability of, robust stability of regular systems to singular systems [6],[7],[8],[9]. Up until now, however, there was little literature concerning the stability problem of singular bilinear systems. The purpose of this paper is to design a state feedback controller for singular bilinear systems such that the resulting closed -loop systems is asymptotically stable.

Notation: For real symmetric matrices X and Y, the notation $X \geq Y$ ($X > Y$) means that the matrix $X - Y$ is positive-semidefinite (positive-define). $\rho(W)$ denotes the spectral radius of the matrix M. $\|x\|, \|A\|$ denote the Euclidean norm of the vector $\|x\|$ and the matrix $\|A\|$, respectively, I_r denotes the identity matrix with dimension $r \times r$.

2. Main results

Consider the following singular bilinear systems

$$E\dot{x} = Ax + Bu + N(x)u \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input , A, B are constant matrices with the appropriate dimensions . The function $N(x) : R^n \to R^{n\times m}$ is a smooth mapping and defined by $N(x) = [N_1x, N_2x, \ldots, N_mx], N_i \in R^{n\times n}, \quad i = 1, 2, \cdots, m$. Also define $B = [b_1, b_2, \cdots, b_m]$.

the system (1) can be written in the following form

$$E\dot{x} = Ax + \sum_{i=1}^{m}(b_i + N_ix)u_i \tag{2}$$

We assume that unforced dynamic system (2) as follows:

$$E\dot{x} = Ax \tag{3}$$

is regular,impulse-free and stable, so that there exist two positive definite matrixes $V and W$ such that

$$E^TVA + A^TVE \leq -E^TWE \tag{4}$$

Now consider the Lyapunov function candidate

$$v(Ex) = (Ex)^TV(Ex) \tag{5}$$

The derivative of v along the trajectories of the bilinear systems (2) is given by

$$\dot{v}(Ex) = x^T(A^TVE + E^TVA)x + 2\sum_{i=1}^{m}u_i(b_i + N_ix)^TVEx$$

By using (4), we have that

$$\dot{v}(Ex) \leq -(Ex)^TW(Ex) + 2\sum_{i=1}^{m}u_i(b_i + N_ix)^TVEx$$

In order that $\dot{v}(Ex)$ is negative, we choose the controller as follows:

$$u_i = -\alpha_i(b_i + N_ix)^TVEx \tag{6}$$

Lemma 1. *For any matrices X, Y, and scalar k, the following inequalities hold:*

$$(i)\ \|X + Y\| \geq \|X\| - \|Y\|$$

$$(ii)\ \|kX\| = |k|\|X\|$$

Theorem 1. *Suppose the systems (3) is regular ,impulse-free and stable, if there exists constant $\gamma > 0$ such that*

$$max_{1\leq i\leq m}\{\|N_ix\|\} \leq \gamma\|Ex\| \tag{7}$$

then the controller (6) can stabilize the systems (2)

Proof: The closed-loop system with the controller (6) can be described as

$$E\dot{x} = Ax - \alpha\Sigma_{i=1}^{m}(b_i + N_ix)(b_i + N_ix)^TVEx \tag{8}$$

From above analysis, we have that $\lim_{t\to\infty} Ex = 0$ for the solution of the closed-loop systems (8). We will show that $\lim_{t\to\infty} x = 0$ for the solution of the closed-loop systems(8). For the systems (3) is regular and free-impulse , there exist two invertible matrixes P and Q such that

$$PEQ = \begin{pmatrix} A_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

Letting $H(x) = (h_1^T(x), h_2^T(x))^T = -P\alpha_i\Sigma_{i=1}^m(b_i+N_ix)(b_i+N_ix)^TVEx$ .If the condition of the Theorem1 holds, then $\lim_{t\to\infty} H(x) = 0$. On the other hand, by taking the transform $x(t) = Qz(t) = Q(z_1^T \ \ z_2^T)$, we have that the closed-loop (6) is equivalent to

$$\dot{z}_1 = A_r z_1 + h_1(Qz) \tag{9}$$

$$0 = z_2 + h_2(Qz) \tag{10}$$

Due to $\lim_{t\to\infty} H(x) = 0$, it is easy to be obtained that $\lim_{t\to\infty} h_2(x) = 0$. From (10), we have that $\lim_{t\to\infty} z_2(t) = 0$. Besides, from $\lim_{t\to\infty} Ex = 0$, we obtained that $\lim_{t\to\infty} z_1 = 0$. Hence, $\lim_{t\to\infty} x = 0$. We notice if $b_i = 0, i = 1, 2, \cdots m$, then system (2) becomes a strictly bilinear control system, described as

$$E\dot{x} = Ax + \sum_{i=1}^{m} N_i x u_i \tag{11}$$

□

Corollary 1. . *Suppose the systems (3) is regular ,impulse-free and stable, if there exists constant $\gamma > 0$ such that*

$$max_{1\le i\le m}\{\|N_ix\|\} \le \gamma\|Ex\| \tag{12}$$

then the controller $u_i = -\alpha_i(N_ix)^TVEx \ \ i = 1, 2, \cdots, m$ can stabilize the systems (11).

It should be pointed out for system (10) the condition (11) is more conservative. To find the new feedback control policy that stabilizes the system (11), let us still consider the Lyapunov function candidate (5) The derivative of v along the trajectories of the bilinear systems (10) is given by

$$\begin{aligned}\dot{v}(Ex) \ &= \ x^T(A^TVE + E^TVA)x \\ &\quad +2\textstyle\sum_{i=1}^m u_i(N_ix)^TVEx \\ &\quad \le -(Ex)^TW(Ex) + 2\textstyle\sum_{i=1}^m u_i(N_ix)^TVEx\end{aligned}$$

In order to ensure that $\dot{v}(Ex) \le 0$, we can choose the following controller

$$u_i = -\alpha_i sign[(N_ix)^TVEx], i = 1, 2, \cdots, m \tag{13}$$

where

$sign(x) = \begin{cases} 1 & , \ x > 0 \\ 0 & , \ x = 0 \\ -1 & , \ x < 0 \end{cases}$ and $\alpha_i > 0, i = 1, 2, \cdots, m$ are constant. Letting $L_E = \{L \in R^{(n-r)\times n} : E^TL = 0, rank[L] + rank[E] = n\}$

Theorem 2. *Suppose the systems (3) is regular, impulse-free and stable, if there exist constant α_i, matrixes $X_i > 0$ and matrixes $E_i \in L, Y_i$ such that*

$$\Sigma_{i=1}^m\alpha_i < m \tag{14}$$

$$\begin{aligned}&A^T(VE + E_iY_i) + (VE + E_iY_i)^TA \\ &+m^2N_i^T(VE + E_iY_i)X_i^{-1}(VE + E_iY_i)^TN_i + X_i < 0\end{aligned} \tag{15}$$

Then the controller(13) can stabilize the strictly bilinear singular systems (11).

Proof: The closed-loop system (11) and(13) is described as

$$E\dot{x} = Ax - \Sigma_{i=1}^m\alpha_i sign[(N_ix)^TVEx]N_ix \tag{16}$$

From the above analyst, we have that $\lim_{t\to\infty} Ex = 0$ for the solution of the closed-loop systems (16). We will show $\lim_{t\to\infty} x = 0$ Let

$$\begin{aligned}&M_i = mN_i, \bar{N}_i := PN_iQ = \begin{pmatrix} N_{i11} & N_{i12} \\ N_{i21} & N_{i22} \end{pmatrix}, \\ &\bar{M}_i := PM_iQ = \begin{pmatrix} M_{i11} & M_{i12} \\ M_{i21} & M_{i22} \end{pmatrix}, \\ &G_i := VE + E_iY_i\end{aligned} \tag{17}$$

It is easy to see that (15) can be rewritten the following form:

$$A^TG_i + G_i^TA + M_i^TG_iX_i^{-1}G_i^TM_i + X_i < 0$$

By the similar procedure as the proof of Theorem1 in [9], we have that $\rho(M_{i22}) < 1$. Hence

$$\rho(N_{i22}) < \frac{1}{m} \tag{18}$$

Let $x(t) = Qz(t) = Q(z_1^T \ \ z_2^T)$. From $\lim_{t\to\infty} Ex = 0$, we obtain $\lim_{t\to\infty} z_1 = 0$. We notice that the closed-loop (16) is equivalent to

$$\dot{z}_1 = A_r z_1 + \Sigma_{i=1}^m u_i N_{i11} z_1 + \Sigma_{i=1}^m u_i N_{i12} z_2 \tag{19}$$

$$0 = z_2 + \Sigma_{i=1}^m u_i N_{i12} z_1 + \Sigma_{i=1}^m u_i N_{i22} z_2 \tag{20}$$

From equation(19), we obtain that

$$\lim_{t\to\infty}(z_2 + \Sigma_{i=1}^m u_i N_{i22} z_2) = 0 \tag{21}$$

On the other hand, From (21) and Lemma1, the following inequality holds

$$\begin{aligned}&\|z_2 + \Sigma_{i=1}^m u_i N_{i22} z_2\| \\ &\ge \|z_2\| - \Sigma_{i=1}^m \alpha_i \|N_{i22}\|\|z_2\| \\ &\ge |z_2\| - \Sigma_{i=1}^m \tfrac{\alpha_i}{m}\|z_2\| \\ &= (1 - \Sigma_{i=1}^m \tfrac{\alpha_i}{m})\|z_2\|\end{aligned} \tag{22}$$

According to (14), (20) and (21), we have that $\lim_{t\to\infty} z_2 = 0$. Hence, $\lim_{t\to\infty} x = 0$ □

3. Numerical Examples

Example1: Consider the singular bilinear systems as follows

$$E\dot{x} = Ax + \Sigma_{i=1}^{3}(b_i + N_i x)u_i \tag{23}$$

where

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, A = \begin{pmatrix} -5 & -2 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} b_2 = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} b_3 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$N_i = K_i E, i = 1, 2, 3, K_i \in R^{n\times n}$$

K_i is a constant matrix. Taking

$$V = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 8 & 1 \\ 0 & 1 & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 8 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

it is easy to verify that(4) and the conditions in Theorem1 are satisfied . Hence, the controller $u_i = -\alpha_i(b_i + N_i x)^T VEx, \quad i = 1, \quad 2, \quad 3$ can stabilize the systems (23).

Example2: Consider the strictly singular bilinear systems as follows

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} u_1 \\ &+ \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} u_2 \end{aligned} \tag{24}$$

We choose that $V = Y_1 = Y_2 = X_1 = X_2 = I, \quad W = 2I$ and $E_1 = E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. According to Theorem2, the controller $u_i = -sign[(N_i x)^T VEx], i = 1, 2$ can stabilize the system (24).

4. Conclusions

This paper studies the stability problem for the singular bilinear systems. First the sufficient conditions are derived, which ensures the existence of state feedback control laws that will stabilize the singular bilinear systems. Then, for a class of singular strictly bilinear systems, we give the method to design a nonlinear controller to stabilize it.

5. Acknowledgement

This work was supported by Korea Research Foundation Grant. (KRF-2002-041-D00363).

References

[1] J.-S. Chiou, Fan-Chu Kung, and Tzuu-Hseng S. Li, "Robust stabilization of a class of singularly perturbed discrete bilinear systems".*IEEE Trans. On Automatic Control*, Vol.45, no.6, pp.1187-1191, 2000.

[2] R. Longchamp, "Stable feedback control of bilinear systems". IEEE Trans. On Automatic control, *IEEE Trans. On Automatic control*, Vol.25, no.2, pp.302-306, 1980.

[3] Hamadi Jerbi , "Global feedback stabilization of new class of bilinear systems". *Systems and Control Letters* , Vol.42, No.4, pp.313-320, 2001.

[4] W. Lin and C.I. Byrnes, "KYP lemma, state feedback and dynamic output feedback in discrete-time bilinear systems". *Systems and Control Letters* , Vol.23, No2.,pp.127-136, 1994.

[5] L. Dai, "Singular control systems". *Berlin, Germany: Springer-Verlag*, 1989.

[6] A. Varga, "On stabilization methods of descriptor systems". *Systems and Control Letters*, Vol.24, no.2, pp.133-138, 1995.

[7] ChonG Lin, Jian Liang Wang, Guang-Hong Yang and James Lam, "Robust stabilization via state feedback for descriptor systems with uncertainties in the derivative matrix". *Int. J. control*, vol.73,No.5, pp.407-415, 2000.

[8] Shengyuan Xu, and ChengWu Yang, "Robust stabilizatin for generalized state-space systems with uncertainty". *Int. J. Contr.*, vol.72, no.18, pp.1659-1664,1999

[9] Shengyuan Xu, Paul Van Dooren, Radu Stefan, and James Lam "Robust stability and stabilization for singular systems with delay and parameter uncertainty". *IEEE Trans. on Automatic Control*, Vol.47,No.7, pp.1122-1128, 2002.