Mixed H_2/H_∞ Control of Two-wheel Mobile Robot

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Abstract: In this paper, we propose a control algorithm for two-wheel mobile robot that can move the rider to his or her command and autonomously keep its balance. The control algorithm is based on a mixed H_2/H_∞ control scheme. In this control problem the main issue is to move the rider while keeping its balance in the presence of disturbances and parameter uncertainties. The disturbance force caused by uneven road surfaces and the uncertainty due to different rider's heights are considered. To this end we first consider a state feedback controller as a basic framework. Secondly, we obtain the state feedback gain K_2 minimizing the H_2 norm and the state feedback gain K_∞ minimizing the H_∞ norm over the whole range of parameter uncertainty. Finally, we select mixed H_2/H_∞ state feedback controller K as the geometric mean of K_2 and K_∞. Simulation results show that the mixed H_2/H_∞ state feedback controller combines the effects of the optimal H_2 state feedback controller and robust H_∞ controller state feedback controller efficiently in the presence of disturbance and parameter uncertainty.

Keywords: Mixed H_2/H_∞ controller, Two-wheel mobile robot, Disturbance attenuation, Parameter uncertainty

1. INTRODUCTION

Two-wheel mobile robot, which behaves like a mobile inverted pendulum, can move forward and backward, while keeping its balance autonomously. If it is to ride a person, a vertical control shaft could be provided for forward and backward motion. On the base chassis of the robot are a pair of DC motors, power amplifiers, batteries and a DSP board with sensors which constitute the controller.

The control aim of two-wheel mobile robot is to move the rider to the commanded direction while autonomously keeping its balance. This system is inherently unstable like an inverted pendulum and is subject to various disturbance forces and parameter uncertainties.

The inverted pendulum control problem has been solved by using various controllers such as linear quadratic regulator (LQR), robust control method and mixed H_2/H_∞ control method. Recently, number of research results about control of inverted pendulum type mobile robot have been reported. Yun-Su Ha et al. [1] used LQR controller for balancing control of the inverse pendulum type self-contained mobile robot. Felix Grasser et al. [2] also used state-space controller for mobile inverted pendulum. Masao Yano et al. [3] applied H_∞ controller for wheeled inverted pendulum and they showed the superiority of H_∞ design compared with a design of conventional pole allocation method. But, the application of mixed H_2/H_∞ controller has not been reported for two-wheel mobile robot.

It is well known that H_2 optimization is to minimize the energy output of a system when the system is faced with white Gaussian noise inputs. It is also clear that the LQR problem is a special H_2 norm minimization problem equivalent to minimizing the 2-norm of the output of a system subject to impulse or disturbance inputs. The H_∞ optimization focuses on the minimization of system output energy to unknown, bounded, energy inputs. The H_∞ control method can be formulated to give robust stability to model uncertainty. Recently, there has been a great deal of interest in formulating a mixed H_2/H_∞ control method which can allow a tradeoff between their objectives [4-7].

In this paper, a mixed H_2/H_∞ controller design method is applied to a two-wheel mobile robot with disturbance and parameter uncertainty. First, we consider a state feedback controller as a basic framework and select a controlled output weighting factor by using LQR control method. Secondly, we obtain the state feedback gain K_2 minimizing the H_2 norm of the closed loop system and the state feedback gain K_∞ minimizing the H_∞ norm of closed-loop system over the range of the parameter uncertainty. Finally, we select the mixed H_2/H_∞ state feedback controller K as the geometric mean of K_2 and K_∞.

In section 2, the system model is described. The brief design method of LQR, H_2 optimal controller, H_∞ optimal controller, mixed H_2/H_∞ controller and the proposed algorithm is presented in section 3. Simulation results are included in section 4 followed by a conclusion.

2. THE SYSTEM MODEL

A two-wheel mobile robot is composed of a base chassis carrying a DC motor, power amplifier and batteries and vertical control shaft. The rider steps on the base chassis and commands the horizontal velocity. Fig. 1 shows the configuration of a two-wheel mobile robot and its coordinates.

The following variables have been chosen to derive dynamic equations.

θ : angle of the rider from the vertical axis
ϕ : angle of the wheel
M : mass of the rider and control shaft

m : mass of the wheel
l : length of center of gravity of the rider and chassis
J_M : moment of inertia of the rider and chassis
J_m : moment of inertia of the wheel
r : radius of the wheel
u : motor torque
g : gravitational force

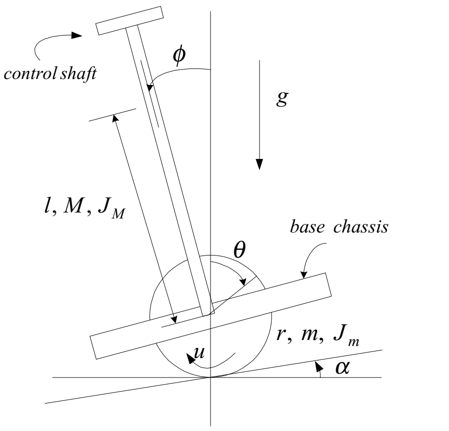

Fig. 1 Configuration of two-wheel mobile robot

The linearized dynamic equations can be written by

$$\begin{bmatrix} J_1 & 0 & -J_3 \\ 0 & 1 & 0 \\ -J_3 & 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & T_g & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w \quad (1)$$

where,

$J_1 = Mr^2 + mr^2 + J_m$

$J_2 = Ml^2 + J_M$

$J_3 = Mlr$

$T_g = Ml\,g$

$w = -(M+m)rg\sin\alpha$

The parameter values in Eq. (1) are given in Table 1. Symbol α represents the inclined slope angle of the surface.

3. MIXED H_2/H_∞ CONTROLLER DESIGN

3.1 H_2, H_∞ and mixed H_2/H_∞ static state feedback controllers

We will consider a linear time invariant system G in the form of Fig. 2 and Eq. (2).

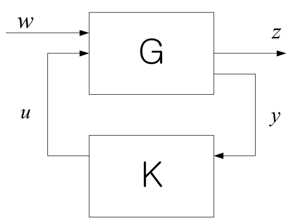

Fig. 2 System framework

$$G := \begin{cases} \dot{x}(t) = Ax(t) + B_2 u(t) + B_1 w(t) \\ z(t) = Cx(t) + Du(t) \\ y(t) = x(t) \end{cases} \quad (2)$$

where x is referred to as the state, u is the control input, w is the exogenous or disturbance input, y is the system output and z is the controlled output. We assume that the state of the system G is available for feedback.

In case of static feedback laws of the form $u(t) = K\,y(t)$, Eq. (3) shows the closed loop system equation.

$$G_{cl} := \begin{cases} \dot{x}(t) = (A + B_2 K)x(t) + B_1 w(t) \\ z(t) = (C + DK)x(t) \end{cases} \quad (3)$$

A controller K is admissible if $A_k \equiv A + BK$ is stable.

H_2 **optimal controller**: The H_2 norm of the closed loop map T_{zw} is given by

$$\|T_{zw}\|_2 = \sqrt{trace(B_1^T P B_1)} = \sqrt{trace(CQC^T)} \quad (4)$$

where P is the observability gramian, which is the solution P of the Lyapunov equation Eq. (5) and superscript T represents the transpose of matrix.

$$(A + B_2 K)^T P + P(A + B_2 K) + (C + DK)^T (C + DK) = 0 \quad (5)$$

equivalently, Q is the controllability gramian, which is the solution of Q of the Lyapunov equation Eq. (6)

$$(A + B_2 K)Q + Q(A + B_2 K)^T + B_1 B_1^T = 0\,. \quad (6)$$

For the system G, the optimal H_2 state feedback is given by $K_2 = -B_2^T X_2$, where X_2 is the stabilizing solution of the following algebraic Riccati equation Eq. (7).

$$X_2 A + A^T X_2 - X_2 B_2 B_2^T X_2 + C^T C = 0. \quad (7)$$

H_∞ **controller**: The H_∞ norm of the closed loop map T_{zw} is given by

$$\|T_{zw}\|_\infty = \sup_\omega \bar{\sigma}\{T_{zw}(j\omega)\} \quad (8)$$

i.e. the H_∞ norm of a system is equal to the supremum of the maximum singular value of its frequency response.

Now by the bounded real lemma, we know that for an admissible K, the closed-loop map T_{zw} has $\|T_{zw}\|_\infty \le \gamma$, if and only if, there exists an $X \ge 0$ such that

$$XA_k + A_k^T X + \gamma^{-2} XB_1 B_1^T X + K^T K + C^T C \le 0\,. \quad (9)$$

The existence of such a state feedback matrix K is equivalent to the existence of a positive semidefinite matrix X that satisfies the Riccati inequality,

$$-XA - A^T X + X(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)X - C^T C \ge 0 \quad (10)$$

Given an X satisfying Eq. (10), $K = -B_2^T X$ is the one such state feedback matrix satisfying Eq. (9). Moreover, the smallest such X is denoted by X_c and is the positive solution to the Riccati equation

$$X_c A + A^T X_c - X_c(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)X_c + C^T C = 0 . \quad (11)$$

The corresponding feedback matrix $K_c = -B_2^T X_c$ is known as the central solution.

Mixed H_2/H_∞ controller: Let S_K be the set of static state feedback matrices that yields a internally stable closed-loop system with $\|T_{zw}\|_\infty \le \gamma$, i.e.,

$$S_K \equiv \{admissible \quad K \mid \|T_{zw}\|_\infty \le \gamma\} \quad (12)$$

where the constant $\gamma > 0$ is given. Then, mixed H_2/H_∞ controller problem is as follows.

Find an internally stabilizing static feedback matrices K that satisfies

$$\min_{K \in S_K} J(K) \quad \text{where} \quad J(K) \equiv trace(B_1^T P B_1) . \quad (13)$$

In other cases, the mixed H_2/H_∞ control problem can be modified to the next problem.

Find controller K that minimizes the following performance index

$$\min \alpha\|T_{zw}\|_2 + \beta\|T_{zw}\|_\infty \quad (14)$$

where α and β are appropriately selected weighting factors [6].

3.2 Application of two-wheel mobile robot

The linearized dynamic equation of two-wheel mobile robot in Eq. (1) can be transformed to state space equation Eq. (15). We select system states $x = [x_1 \quad x_2 \quad x_3] = [\dot{\theta} \quad \phi \quad \dot{\phi}]$ and controlled outputs $z = [c\dot{\theta} \quad u]^T$.

$$\dot{x} = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & 0 & 1 \\ 0 & a_2 & 0 \end{bmatrix} x + \begin{bmatrix} b_{21} \\ 0 \\ b_{22} \end{bmatrix} u + \begin{bmatrix} b_{11} \\ 0 \\ b_{12} \end{bmatrix} w \quad (15)$$

$$u = Kx = [k_1 \quad k_2 \quad k_3]x \quad (16)$$

$$z = \begin{bmatrix} c\dot{\theta} \\ u \end{bmatrix} = \begin{bmatrix} c & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} x \quad (17)$$

where $a_1 = \frac{J_3}{J_4^2} T_g$, $a_2 = \frac{J_1}{J_4^2} T_g$, $b_{21} = \frac{J_2 + J_3}{J_4^2}$, $b_{22} = \frac{J_1 + J_3}{J_4^2}$, $b_{11} = \frac{J_2}{J_4^2}$, $b_{12} = \frac{J_3}{J_4^2}$, $J_4^2 = J_1 J_2 - J_3^2$,

c: *constant weighting factor*

In this paper, we consider the parameter uncertainty such that the length of center of gravity of the rider and chassis l can be varied 50[%] from nominal value and disturbance w caused by the inclined slope α . In general, H_2 optimization is to minimize the energy output of a system when the system is faced with white Gaussian noise inputs. It is also clear that the LQR problem is a special H_2 norm minimization problem equivalent to minimizing the 2-norm of the output of a system subject to impulse or disturbance inputs. On the other hand, the H_∞ optimization focuses on the minimization of system output energy to unknown, bounded, energy inputs. The H_∞ control can be formulated to give robust stability to model uncertainty. Recently, there has been a great deal of interest in formulating a mixed H_2/H_∞ control method which can allow a tradeoff between their objectives.

We suggest the mixed H_2/H_∞ controller design algorithm that the mixed H_2/H_∞ controller gain matrix K is determined as the geometric mean of K_2 and K_∞ instead of using the performance index in Eqs. (13)~(14) .

Algorithm:

Step 1: We select the controlled output weighting factor c by using LQR controller design method. In this step, we consider the tracking error for reference wheel angular velocity.

Step 2: After selecting the proper weighting factor c , we find the H_2 optimal controller gain matrix K_2 that minimize the H_2 norm varying K over the entire parameter range of length l .

Step 3: Similarly, we find the H_∞ optimal controller gain matrix K_∞ that minimize the H_∞ norm varying H_∞ norm varying K over the entire parameter range of length l .

Step 4: Determine the mixed H_2/H_∞ controller gain matrix K that is the geometric mean of K_2 and K_∞ .

4. SIMULATION

4.1 System parameters

System parameters are for the simulation as follows. The length l is considered as a parameter uncertainty.

Table 1 System Parameters

Symbol	Value	Symbol	Value
M	80 $[kg]$	J_M	70 $[kgm^2]$
m	0.8 $[kg]$	J_m	0.01 $[kgm^2]$
l	0.4~1.2 $[m]$	r	0.15 $[m]$
g	9.8 $[m/s^2]$	α	0.2 $[rad]$

4.2 Results

To select the controlled output weighting factor c of state $\dot{\phi}$ in Eq. (23), we obtain the LQR optimal state feedback gain matrix K_L for nominal length $l = 0.8[m]$ while varying weighting factor c . In this LQR control design, the weighting matrices Q and R of the performance index in Eq. (18) are presented in Eq. (19).

$$J = \frac{1}{2}\int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t))dt , \quad (18)$$

$$Q = \begin{bmatrix} c^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = 1. \tag{19}$$

The steady state ramp input error for wheel angular velocity $\dot{\phi}$ while varying weighting factor c is shown in Fig. 3. We determined the steady state error 12 % as a decision value and obtained $c = 13$ with LQR controller gain K_L =[-13, 813, 348].

In order to obtain the H_2 optimal gain matrix K_2 in step 2, we draw the H_2 norm curves of the closed-loop system varying k_1 and fixing k_2, k_3 alternately. We can know that the H_2 norm has monotonously increasing or decreasing characteristic according to varying length l. Therefore, it is not need to search over the entire range of length l but both sides of $l = 0.4[m]$ and $l = 1.2[m]$. By iteratively varying k_1, k_2, and k_3, we can find the H_2 optimal gain matrix K_2=[-11.7, 1078, 333] minimizing the maximum H_2 norm of the closed loop system.

By the same method as obtaining K_2, we obtain H_∞ optimal gain matrix K_∞ =[-16.8, 845, 484].

Finally, we determine the mixed H_2 / H_∞ gain matrix K based on the obtained K_2 and K_∞ in step 2 and step3. As mentioned in step 4, we select the K =[-14, 954, 401] by using the geometric mean of K_2 and K_∞ as follows.

$$k_i = \sqrt{k_{2i} \cdot k_{\infty i}} \quad (i = 1,\ 2,\ 3).$$

Figs. 4~5 show the H_2 norms and H_∞ norms of closed systems, respectively, obtained from the three control schemes; H_2, H_∞ and mixed H_2 / H_∞ controller. We see that the H_2 norm of the mixed H_2 / H_∞ controller is smaller than the H_2 norm of H_∞ controller in both ends of range l that represent the worst cases. The H_∞ norm of mixed H_2 / H_∞ controller is smaller than the H_∞ norm of H_2 controller in both ends of range l. This indicates the positive effects of mixing H_2 and H_∞ controllers.

Figs. 6~7 show the transient responses of wheel angular velocity $\dot{\theta}$, angle of the rider from vertical axis, ϕ, and the control input u and the frequency response of controlled output z for the three different controllers with nominal length l =0.8 [m]. We can see from Fig. 4 and Fig. 5 that for nominal length l, the responses are almost the same because the differences of the H_2 and H_∞ norm between three controllers are small.

Figs. 8~9 show responses for the case of l =0.4 [m]. On the other hand, Figs. 10~11 show responses for the case of l =1.2 [m]. It is shown that the mixed H_2 / H_∞ controller has the advantage of compensating the H_2 optimal controller and robust H_∞ optimal controller in the presence of the disturbance due to inclined slope α and parameter uncertainty of length l. In all of simulations, the initial state is $x_0 = [\dot{\theta}_0, \phi_0, \dot{\phi}_0]^T$ =[-1, 0.1, 0.1]^T.

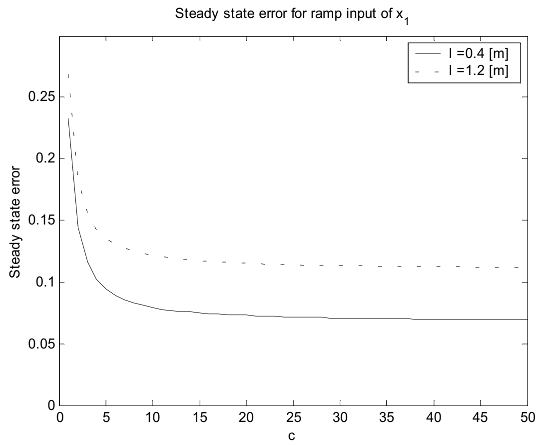

Fig. 3 Steady state error for ramp input of $\dot{\theta}$

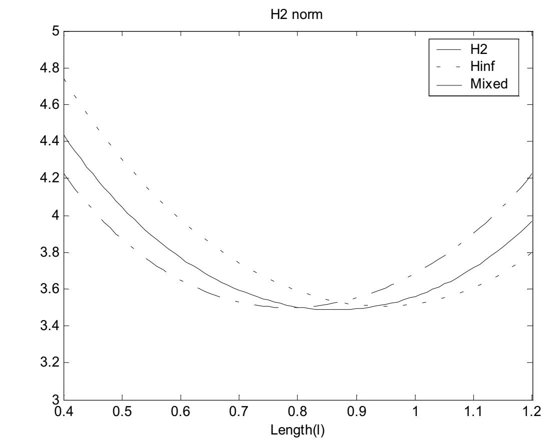

Fig. 4 H_2 norm of closed loop system for varying l

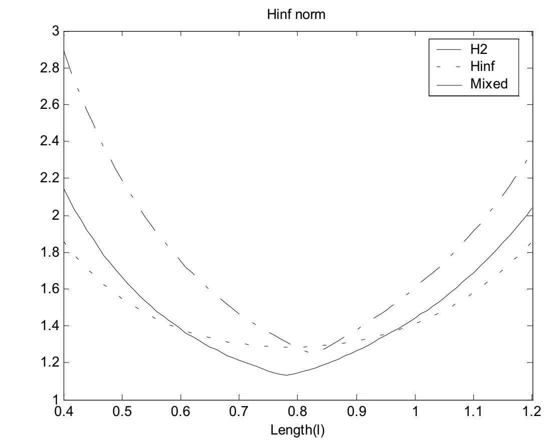

Fig. 5 H_∞ norm of closed-loop system according to varying l

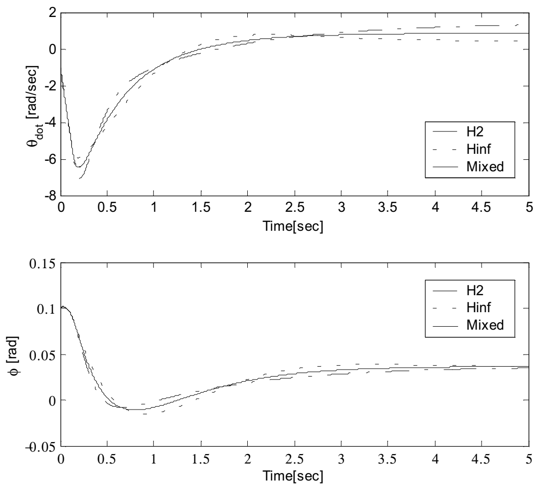

Fig. 6 Transient responses in case of $l = 0.8[m]$ (Top: $\dot{\theta}$, Bottom: ϕ)

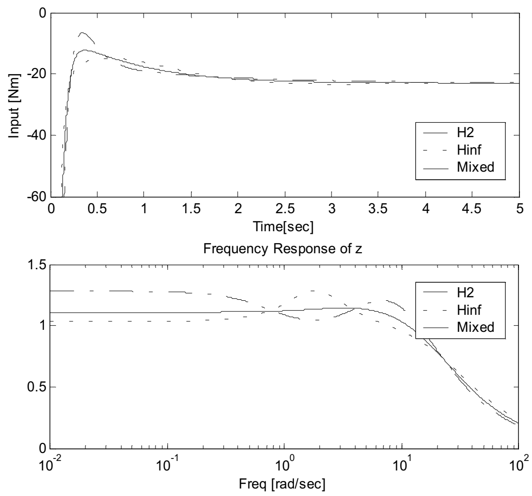

Fig. 7 Top: transient response of u, Bottom: frequency response of z in case of $l = 0.8[m]$

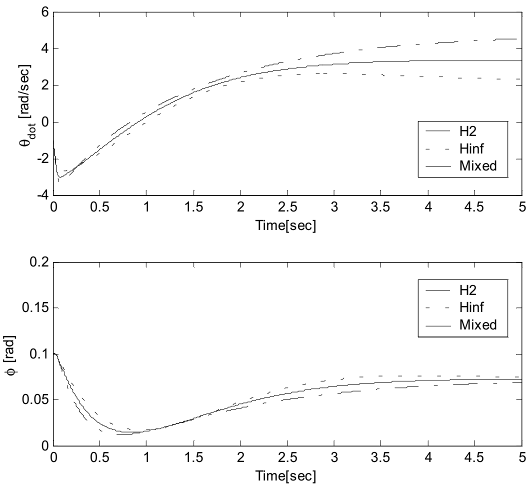

Fig. 8 Transient responses in case of $l = 0.4[m]$ (Top: $\dot{\theta}$, Bottom: ϕ)

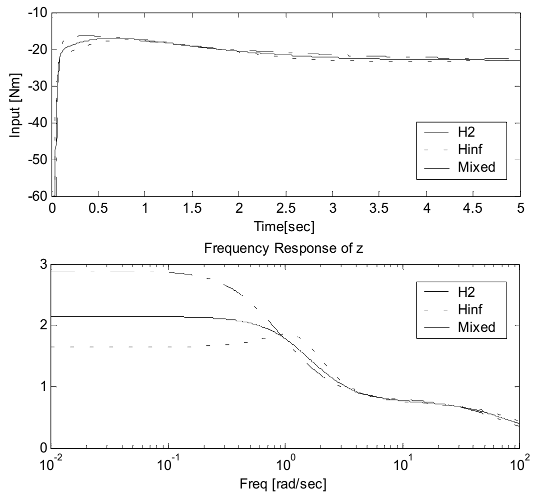

Fig. 9 Top: transient response of u, Bottom: Frequency response of z in case of $l = 0.4[m]$

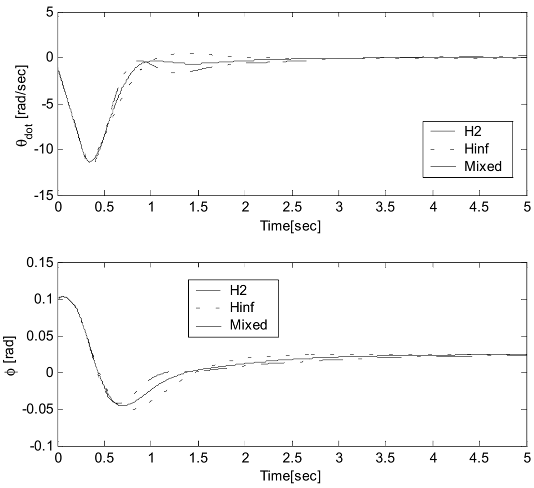

Fig. 10 Transient responses in case of $l = 1.2[m]$ (Top: $\dot{\theta}$, Bottom: ϕ)

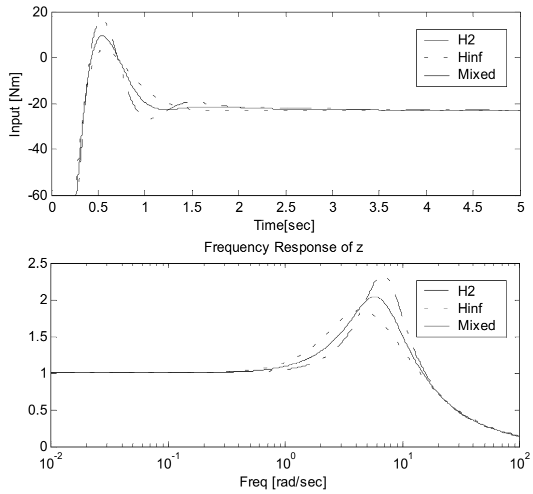

Fig. 11 Top: transient response of u, Bottom: frequency response of z in case of $l = 1.2[m]$

5. CONCLUSION

This paper is concerned with a mixed H_2/H_∞ state feedback controller design for the two-wheel mobile robot that is subject to disturbance and parameter uncertainty. We propose the algorithm to design a mixed H_2/H_∞ state feedback controller.

We suggest the mixed H_2/H_∞ controller design algorithm that the mixed H_2/H_∞ controller gain matrix K is determined as the geometric mean of the H_2 optimal controller gain matrix K_2 and the H_∞ optimal controller gain matrix K_∞. Simulation results show that the proposed H_2/H_∞ control method combines the desirable effects of the optimal H_2 controller and the robust H_∞ controller for the system with disturbance and parameter uncertainty.

REFERENCES

[1] Yun-Su Ha and Shin'ichi Yuta, "Trajectory tracking control for navigation of the inverse pendulum type self-contained mobile robot," *Robotics and Autonomous Systems,* Vol. 17, pp. 65-80, 1996.

[2] Felix Grasser, Aldo D'Arrigo, Silvio Colombi and Alfred C. Rufer, "JOE: A Mobile, Inverted Pendulum," *IEEE Trans. on Industrial Electronics*, Vol. 49, No. 1, pp. 107-114, 2002.

[3] Masao Yano, Satoru Kuramochi and Hidetoshi Ikeda, "Evaluation of the Robustness of H_∞ Control Design versus Pole Allocation Control Design Using of a Wheeled Inverted Pendulum," *The 5th International Workshop on Advanced Motion Control*, pp. 59-63, 1998.

[4] Pramod P. Khargonekar and Mario A. Rotea, "Mixed H_2/H_∞ Control: A Convex Optimization Approach," *IEEE Trans. on Automatic Control*, Vol. 36, No. 7 pp. 824-837, 1991.

[5] B. Halder, B. Hassibi and T. Kailath, "Design of Optimal Mixed H_2/H_∞ Static State Feedback Controllers," *Proc. of the American Control Conference*, pp. 3239-3243, 1998

[6] M. Dettori, V. Prodanovic and C. W. Scherer, "Mixed objectives MIMO control design for a Compact Disc Player," *Proc. of the American Control Conference*, pp. 1284-1288, 1998

[7] Michael Trentini and Jeff K. Pieper, "A Case Study of Multi-Objective Control: Synthesis and Analysis," *Proc. of the 1998 IEEE International Conference on Control Applications,* pp. 1001-1005, 1998