AN LMI APPROACH TO AUTOMATIC LOOP-SHAPING OF QFT CONTROLLERS

Vaheed S. Bokharaie*, Ali Khaki-Sedigh+

** Graduate Student, email:* v_samadibokharaie@ee.kntu.ac.ir
+ Full Professor, email: sedigh@eetd.kntu.ac.ir

K.N. Toosi University of Technology, Tehran, Iran

Key Words: Optimization, Quantitative Feedback Theory (QFT), Convex Programming, Linear Matrix Inequality (LMI), Loop-Shaping

Abstract

Quantitative Feedback Theory (QFT) is one of effective methods of robust controller design. In QFT design we can considers the phase information of the perturbed plant so it is less conservative than H_∞ and μ-synthesis methods and as be shown, it is more transparent than the sensitivity reduction methods mentioned .

In this paper we want to overcome the major drawback of QFT method which is lack of an automatic method for loop-shaping step of the method so we focus on the following problem: Given a nominal plant and QFT bounds, synthesize a controller that achieves closed-loop stability and satisfies the QFT boundaries. The usual approach to this problem involves loop-shaping in the frequency domain by manipulating the poles and zeros of the nominal loop transfer function. This process now aided by recently developed computer aided design tools proceeds by trial and error and its success often depends heavily on the experience of the loop-shaper. Thus for the novice and First time QFT user, there is a genuine need for an automatic loop-shaping tool to generate a first-cut solution. Clearly such an automatic process must involve some sort of optimization, and while recent results on convex optimization have found fruitful applications in other areas of control theory we have tried to use LMI theory for automating the loop-shaping step of QFT design.

1. Introduction

Quantitative Feedback Theory (QFT) is one of effective methods of robust controller design. Particularly it allows us to obtain controllers less conservative than other robust controller design methods like H_∞ and μ-synthesis. One feature that distinguishes QFT from other frequency-domain methods, such as H_∞ and LQG/LTR, is its ability to deal directly with uncertainty models and robust performance criteria. This is achieved by translating robust performance specifications and uncertainty models into so-called QFT bounds. These bounds, typically displayed on a Nichols chart-like plot, then serve as a guide for shaping the nominal loop transfer function which involves the manipulation of gain, poles and zeros. This design process is executed efficiently using computer aided design software and is effective for “simple” problems. Nevertheless, QFT designers are often challenged by such control problem; due to a lack of loop- shaping experience, and could benefit from an algorithm that automatically provides a first-cut solution to the loop-shaping problem. In addition, an automatic loop-shaping facility would enhance the capabilities of the expert QFT designer. Automatic loop-shaping algorithms have been proposed over the past twenty years and this paper reports on a new version.

In recent years, convex programming and LMI theory has been used widely in solving some important control problems, so we tried to make use of these methods for solving QFT loop-shaping problem for the very first time.

2. The QFT Design Technique

The general QFT problem is how to design controller C(s) and pre-filter F(s) such that for a given set of uncertain plants $P \in \{P\}$ with perturbed parameters $\alpha \in \Omega$ the following specifications are satisfied:

(i) Robust Stability:

$$T_R = FT = \frac{FCP}{1+CP} = \frac{FL}{1+L}$$

must be exponentially stable $\forall \alpha \in \Omega$.

*(ii) Robust tracking performance: T*wo time functions a(t) and b(t), are given and a command input r(t) (for example a

step function) that specify the output tolerance of y(t) in the form:

$$a(t) \le y(t) \le b(t) \qquad \forall P \in \{P\} \qquad (1)$$

These tracking specifications in the time domain can be translated into the frequency domain upper and lower bounds for $T_R(j\omega)$, as shown in figure 2, that satisfies:

$$a(\omega) \le |T_R(j\omega)| \le b(\omega) \quad \textit{in dB units} \qquad (2)$$

(iii) Output disturbance rejection specification: A function *D(ω)* is given that specifies the output specifications of $T_d(j\omega)$, in then form:

$$|T_d(j\omega)| \le D(\omega) \qquad \forall P \in \{P\} \qquad (3)$$

where $T_d(s) = \dfrac{y_d(s)}{d(s)} = \dfrac{1}{1 + C(s)P(s)} \equiv S(s)$

and *d(s)* is the output disturbance function.

(iv) *Input disturbance rejection specification:* A function *D'(ω)* is given that specifies the output specifications of $T'_d(j\omega)$, in then form:

$$|T'_d(j\omega)| \le D'(\omega) \qquad \forall P \in \{P\} \qquad (4)$$

where $T'_d(s) = \dfrac{y_d(s)}{d'(s)} = \dfrac{P(s)}{1 + G(s)P(s)} = S(s).P(s)$

and *d'(s)* is the output disturbance function.

In the classical QFT design, the above specifications will be transformed into boundaries for some pre-specified frequencies (called trial frequencies) in Nichols chart and we have to derive an open-loop transfer function $L_0(s) = P_0(\alpha_0, s).C(s)$ such that L_0 lie above the boundaries of all frequencies (note that $P_0(\alpha_0, s)$ is the nominal plant). In this paper we will introduce an LMI based method for automating this process. Here we focus on the loop-shaping problem (finding a suitable C). The pre-filter F(s) can then be calculated easily.

3. LMI Theory

Consider the problem of minimizing a linear function of a variable $x \in \Re^n$ subject to a matrix inequality:

Minimize $c^T x$
Subject to $F(x) \ge 0$ (5)

Where $F(x) \hat{=} F_0 + \sum_{i=1}^{m} x_i F_i$.

The problem data are the vector $c \in \Re^n$ and m+1 symmetric matrices F_0 , ..., $F_m \in \Re^{nxn}$ The inequality sign in $F(x) \ge 0$ means that F(x) is positive semi-definite, i.e., $z^T F(x) z \ge 0$ for all $z \in \Re^n$. We call the inequality $F(x) \ge 0$ a linear matrix inequality and the above problem a semi-definite program or LMI problem. It is also called a convex optimization problem since its objective and constraints are convex:

If $F(x) \ge 0$ and $F(y) \ge 0$ then for all λ, $0 \le \lambda \le 1$ then $F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y)$.

For example, linear programming problem:

Minimize $c^T x$
Subject to $Ax + b \ge 0$

is a LMI problem.

The most attractive feature of LMI theory is that LMI problems just have global minimums and there are no local minimums. Another important feature is that we can convert various LMI problems into a single LMI problem. Suppose that we have k LMIs $F_i(x) \ge 0$ (i=1,..,k). These k LMIs are equivalent to the LMI $F(x) \ge 0$ in which:.

$$F(x) = \begin{bmatrix} F_1(x) & \underline{O} & \dots & \underline{O} \\ \underline{O} & F_2(x) & \dots & \underline{O} \\ \dots & \dots & .. & \dots \\ \underline{O} & \underline{O} & .. & F_k(x) \end{bmatrix}$$

In general, LMI problems don't have analytical solutions but there are a lot of efficient numerical methods (like interior point method) for solving these problems, so all we have to do is to transform our optimization problem into one of standard LMI problems that has been mentioned in [5].

4. Automatic Loop-Shaping

In this section we will show how to convert the QFT loop-shaping problem into an LMI problem. We first start with the robust tracking specification. As mentioned above, we want to find a controller C(s) such that:

$$a(t) \le y(t) \le b(t) \quad \forall \alpha \in \Omega, \forall t \ge 0 \qquad (6)$$

or equivalently, so that:

$$\left(y(t) - \frac{a(t)+b(t)}{2}\right)^2 \le \left(\frac{b(t)-a(t)}{2}\right)^2 \quad \forall \alpha \in \Omega, \forall t \ge 0 \qquad (7)$$

An exact frequency-domain equivalent to the above time-domain inequality is unknown. A slightly weaker condition to

$$\int_0^\infty \left(y(t) - \frac{a(t)+b(t)}{2} \right)^2 \le \int_0^\infty \left(\frac{b(t)-a(t)}{2} \right)^2 \quad \forall \alpha \in \Omega, \forall t \ge 0 \quad (8)$$

It is noted that the integration operation in (8) converts the original L_∞ (amplitude) into constraints to an L_2 (energy) constraint. Therefore satisfaction of (8) does not necessarily imply satisfaction of (7). Here a(t) and b(t) are upper and lower step-response specifications respectively. Also y(t) is the closed-loop response output as a function of the plant parametric uncertainty vector $\alpha \in \Omega$. A rigorous frequency-domain translation of bounds such as in (8) can be made negligible [3]. It will be shown that the relaxed problem is roughly equivalent to the solution of a frequency-domain sensitivity reduction problem. Let:

$$\frac{a(t)+b(t)}{2} = y_0(t) \;, \; \frac{b(t)-a(t)}{2} = \upsilon(t) \quad \forall \alpha \in \Omega, \forall \omega \quad (9)$$

The by Parseval's theorem, a sufficient condition for satisfaction of (8) is:

$$\left| y(j\omega) - y_0(j\omega) \right| \le \left| v(j\omega) \right| \quad \forall \alpha \in \Omega, \forall \omega \quad (10)$$

We assume that a(t),b(t) and hence y(t) are all Laplace transformable. Because of internal stability, every y(t) is required to have the same number N_z of non-minimum-phase zeros. Bode's sensitivity theorem [Bod1945], normalized with respect to y_0 shows that (temporarily dropping the argument jω:

$$\frac{y - y_0}{y_0} = \frac{FT - FT_0}{FT_0} = S \frac{P - P_0}{P_0} \quad (11)$$

where:

$P(s) = P(\alpha, s)$ is uncertain plant,

$P_0(s) = P(\alpha_0, s)$ is nominal plant,

$L_0(s) = P_0(\alpha, s).C(s)$ is nominal open-loop transfer function,

$L(s) = P(\alpha, s).C(s)$ is open-loop transfer function,

C(s) is the controller, and also:

$$S(s) = \frac{1}{1+L(s)}, \; T(s) = \frac{L(s)}{1+L(s)} \;, T_R(s) = F(s) \frac{L(s)}{1+L(s)}$$

Since the tracking specification imposes the constraint $\left| y - y_0 \right| \le \left| \upsilon \right|$, we have from (10) and (11) that:

$$\left| S(j\omega) y_0(j\omega) \delta_{P_0}(j\omega) \right| \le \left| \upsilon(j\omega) \right| \quad (12)$$

is implied by:

$$\max_{p \in P} \left| S(j\omega) \right| \le \left| \frac{\upsilon(j\omega)}{y_0(j\omega) \delta_{P_0}(j\omega)} \right| \quad \forall \alpha \in \Omega, \forall \omega \in \Re \quad (13)$$

where: $\delta_{P_0}(\omega) \equiv \max_{p \in P} \left| \frac{P(j\omega) - P_0(j\omega)}{P_0(j\omega)} \right|$

Satisfaction of inequality (13) is a sufficient condition for the L_2 tracking specification to be met. Note that we can use equation 13 independent of the method we used for deriving $y_0(j\omega)$ and $\upsilon(j\omega)$. In fact QFT designers has understood that finding the frequency domain equivalents of time domain specifications is not so difficult and for almost every practical time domain specifications we can find a 2nd order frequency domain equivalent, so we can ignore equation (8) and obtain $y_0(j\omega)$ and $\upsilon(j\omega)$ using every method we prefer and then equation (13) can be used.

Now the robust tracking specification can be stated as follows:

$$\left| S(j\omega) \right| \le \frac{\left| \upsilon(j\omega) \right|}{\left| y_0(j\omega) \right| \left| \delta_{P_0}(j\omega) \right|} \equiv M_T(\omega) \quad \forall \alpha \in \Omega, \forall \omega \quad (14)$$

We also mentioned that robust output disturbance rejection specification is as follows:

$$\left| S(j\omega) \right| = \left| \frac{1}{1 + L(j\omega)} \right| \le M_D(\omega) \quad \forall \alpha \in \Omega, \forall \omega \quad (15)$$

And we also want to have robust stability. It means that:

$$T_R(s) = F(s) \frac{C(s)P(s)}{1 + C(s)P(s)} \quad (16)$$

must be exponentially stable $\forall \alpha \in \Omega$. It means that the maximum gain of T_R must be less than a priori defined value. The pre-filter *F(s)* is always a stable and is almost always a low pass filter, so its maximum gain is almost negligible and we can interpret this specification as finding controller *C(s)* such that the high frequency gain of *T(s)* will be less than a defined value. We can easily this specification into a limit over the maximum gain of sensitivity function *S(s)* Hence the robust stability specification can be stated as follows:

$$\left| S(j\omega) \right| \le M_S(\omega) \quad \forall \alpha \in \Omega, \forall \omega \quad (17)$$

in which M_S is specified by designer.

Now we must find a controller C(s) such that equations (15),(16),(17) be satisfied simultaneously, so we define:

$$M(\omega) \equiv \min\{M_D(\omega), M_T(\omega), M_S(\omega)\} \in L_\infty \quad (18)$$

Then the above constraints are simultaneously satisfied if:

$$|S(j\omega)| \le M(\omega) \qquad \forall \alpha \in \Omega, \forall \omega \quad (19)$$

Therefore, our aim is to find a controller *C(s)* such that inequality (19) will be satisfied. To do this, we define a real-rational function *W(jω)* as upper bound for *1/M(ω),* so (19) converts to:

$$|S(j\omega)| \le \frac{1}{W(j\omega)} \qquad \forall \alpha \in \Omega, \forall \omega$$

or:

$$|S(j\omega)W(j\omega)| \le 1 \qquad \forall \alpha \in \Omega, \forall \omega$$

or equivalently:

$$\|S(j\omega)W(j\omega)\|_\infty \le 1 \qquad \forall \alpha \in \Omega \quad (20)$$

Note that this is not an ordinary sensitivity reduction problem because here *S(jω)* is perturbed sensitivity function, not nominal sensitivity function. Using similar procedure, we can transform the input disturbance rejection specification into the below inequality:

$$\|T(j\omega)W_T(j\omega)\|_\infty \le 1 \qquad \forall \alpha \in \Omega \quad (21)$$

in which *T(jω)* is perturbed complementary sensitivity function. Because both inequalities must be satisfied simultaneously, we must find a controller C(s) such that:

$$\left\| \begin{matrix} S\,W \\ T\,W_T \end{matrix} \right\|_\infty \le 1 \qquad \forall \alpha \in \Omega \quad (22)$$

The above problem is equivalent to infinite nominal sensitivity reduction problem and instead of solving it, we consider just finite numbers of uncertain parameters and convert the problem to a finite number of sensitivity reduction problems. For example if parameter α_1 varies between 1 and 2 we can consider just 1, 1.5, 2 (or more values depending on the problem). In [1] a method for converting the sensitivity reduction into LMI problem has been introduced. We can used this method and change all the finite sensitivity reduction problems into LMI form and as mentioned in section 3 all of the simultaneous LMI problems can be transformed into one LMI problem and can be solved by the available packages like MATLAB LMI toolbox .

Note that using this method, the loop-shaping problem has been automated and there is no need for calculating and plotting the QFT boundaries in Nichols chart or complex plane.

5. Results

The following example is developed from a flight control problem due to [7]. The problem is to design outer loop compensation for a basic longitudinal autopilot. The model given is for a four-engine jet transport of unspecified type in the landing configuration. The longitudinal motion of the aircraft in open-loop has the basic structure

$$P(\alpha, s) = \frac{k\left(1 + \frac{s}{z}\right)}{s\left(1 + \frac{s}{p}\right)\left(1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n{}^2}\right)}$$

From aerodynamic data supplied by [7], appropriate ranges for the uncertain parameters are identified. This information is summarized in table 1:

Parameter	Nominal Value	Range
k	2.0	[0.2,2.0]
z	0.5	[0.5,0.75]
p	10.0	[1.0,10.0]
ω_n	6.0	[5.0,6.0]
ζ	0.8	[0.8,0.9]

Table 1

Lower performance boundary a(ω) and upper performance boundary b(ω) are chosen to be:

$$a(\omega) = \left| \frac{1}{(1 + j\omega)(1 + j\omega)(1 + 0.5j\omega)} \right|$$

$$b(\omega) = \left| \frac{1 + j\omega/0.35}{(1 + j\omega/0.5)(1 + j\omega/3)} \right|$$

Design frequencies are chosen as [0.01, 0.05, 0.1, 0.2, 1.0, 5.0, 10.0].

Using the MATLAB LMI toolbox, the controller was determined to be:

$$C(s)=\frac{3.355\left(\frac{s}{0.630}+1\right)\left(\frac{s}{2.994}+1\right)\left(\frac{s}{2.994}+1\right)\left(\frac{s}{7.035}+1\right)\left(\frac{s}{7.035}+1\right)}{\left(\frac{s}{0.714}+1\right)\left(\frac{s}{0.822}+1\right)\left(\frac{s}{3.640}+1\right)\left(\frac{s}{20.254}+1\right)\left(\frac{s}{714.29}+1\right)}$$

The high frequency gain of $L_0(s)$ is reduced to $5.344(10^5)$ which is 55% less than the expert's design [6] and accordingly over-design in the loop was essentially eliminated.
In order to meet the closed-loop tracking requirement in the two degree of freedom structure, a suitable pre-filter must be obtained. This is designed to the methodology described in [4]. This pre-filter is given as:

$$F(s)=\frac{(1+s/0.5)}{(1+s/0.6)(1+s/2.5)}$$

A collection of closed-loop Bode plots for the extreme plant parameter conditions as described in the above table are given in Fig. 1. Although there is no direct implication or time domain performance, it is seen that the corresponding time response are favorable, as shown in figure 2.

6. Conclusions:

In this paper a method for automatic loop-shaping of QFT controllers has been introduced for the first time. The design process has been converted to an LMI problem which can be solved using efficient numerical methods. The results show the effectiveness of this new method.

References:

[1] Gahinet P. and Apkarian P., *The Linear Matrix Inequality Approach to H_∞ Control,* Int. J. Robust and Nonlinear Control, 4, 421-448, (1994)

[2] Bode H., *Network Analysis and Feedback Amplifier Design,* Van Nostrand, New York, (1945)

[3] Krishnan K.R. and Cruikshanks A., *Frequency Domain Design of Feedback Systems for Specified insensitivity for time-domain response to parameter variation,* Int. J. of Control, 25, 609-620 (1977)

[4] Houpis C.H. and Rasmussen S.J., *Quantitative Feedback Theory, Fundamentals and Application*, Marcel-Dekker, New York (1999)

[5] Boyd S., El Ghaoui L., *Convex Optimization,* Draft Copy, ver. 12/03

[6] Bailey F.N., Cockburn J.C., Loop Gain Phase Shaping for a Single Input Single Output Robust Controllers, Proceeding of ACC, Boston, (1991).

[7] Blakelock, J.H., 1965, *Automatic Control of Aircraft and Missiles,* Wiley, Newyork.

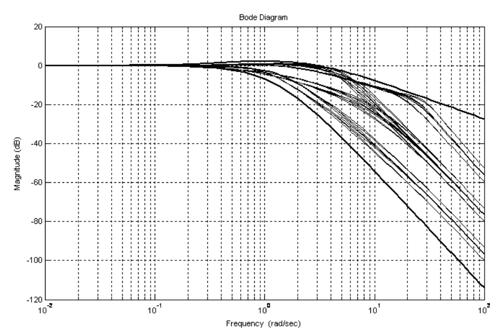

Figure 1

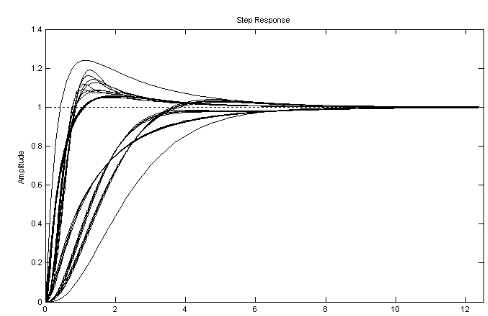

Figure 2