Frequency Weighted Model Reduction Using Structurally Balanced Realization

Do Chang Oh* and Jong Hae Kim**

*Department of Control and Instrumentation Engineering, Konyang University, Nonsan, Korea
(Tel: +82-41-730-5369; Fax: +82-41-736-4079; Email:docoh@konyang.ac.kr)
**Division of Electronics, Information and Communication Engineering,Sunmoon University, Asan, Korea
(Tel: +82-41-530-2352; Email:kjhae@sunmoon.ac.kr)

Abstract: This paper is on weighted model reduction using structurally balanced truncation. For a given weighted(single or double-sided) transfer function, a state space realization with the linear fractional transformation form is obtained. Then we prove that two block diagonal LMI(linear matrix inequality) solutions always exist, and it is possible to get a reduced order model with guaranteed stability and *a priori* error bound. Finally, two examples are used to show the validity of proposed weighted reduction method, and the method is compared with other existing methods.

Keywords: Model reduction, frequency weighted reduction, balanced realization, error bound

1. Introduction

Balanced truncation and Hankel-norm approximation are based on the balanced realization using solutions of Lyapunov equations, and frequency domain error bounds can be obtained from the Hankel singular values[1],[2]. Enns [3] extended the balanced truncation technique to the frequency weighted case. With only input or output weighting (single-sided weighting) present, stability of the reduced order model is guaranteed. However, with both weighting(double-sided weighting) functions present, Enns' method may yield unstable models for stable original systems. To overcome the drawback of instability, Lin and Chiu [4] proposed a new frequency weighted balanced reduction technique, and Sreeram *et. al.*[5] extended it to the general case. Frequency domain error bound formulas for the above two methods were proposed by Kim [6] and Sreeram [5], respectively. To overcome the drawback that the computation of error bounds is achieved iteratively, Wang *et. al.*[7] proposed another frequency weighted model reduction technique, and derived a formula for the error bound. Although Wang's method solves the problem that the error bounds are calculated iteratively, it has two problems to the author's knowledge. One is that the method still does not solve the problem that the error bound formula involves computation of infinity-norms of low order transfer functions. The other is that it has to go through many complicated steps to find the reduced order model. In this paper, we propose a simple frequency weighted model reduction using structurally balanced truncation(SBT) and show the existence of the solution, which is obtained via two linear matrix inequalities(LMIs). This method was first mentioned by Zhou *et. al.*[8]. To find a reduced order model, we prove that two block diagonal LMI solutions always exist. Furthermore, a simple *a priori* error bound is derived without any computation of infinity-norm, some examples are used to show the validity of proposed method, and the method is compared with other existing methods. Now, the notation used in this paper is introduced. M^T denotes the transpose of matrix $M \in \Re^{p\times q}$ and $tr(M)$, if $p = q$, denotes the trace of M, and $\lambda_i(M)$ denotes ith eigenvalue of M. A diagonal matrix with $m_1, m_2, \cdots, m_p$ as its diagonal elements is denoted by $diag(m_1, m_2, \cdots, m_p)$. Similarly, a block diagonal matrix with $M_1, M_2, \cdots, M_k$ as the block diagonal entries is denoted by $diag(M_1, M_2, \cdots, M_k)$. Let $P(s)$ be a rational proper transfer matrix, the set of all rational proper and stable transfer function is denoted by RH_∞.

2. Structurally balanced reduction

In this section, we review the structurally balanced reduction proposed by Zhou [8]. Consider the nth-order transfer function $P(s)$ and the mth-order transfer function $K(s) =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $P(s)$ is partitioned as

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}. \quad (1)$$

The closed-loop transfer function in linear fractional transformation form is given by

$$T_{zw} = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} =: \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right], \quad (2)$$

where $K(s)$ is a high order model, and $T_{zw}(s) \in RH_\infty$. If there exist block diagonal solutions M and N such that

$$\bar{A}M + M\bar{A}^T + \bar{B}\bar{B}^T \le 0, \quad M = diag(M_p, M_k) \ge 0, \quad (3)$$
$$\bar{A}^TN + N\bar{A} + \bar{C}^T\bar{C} \le 0, \quad N = diag(N_p, N_k) \ge 0, \quad (4)$$

then nonsingular matrices T_p and T_k exist such that

$$\begin{aligned} T_pM_pT_p^T &= T_p^{-T}N_pT_p^{-1} \\ &= diag(\zeta_1, \zeta_2, \cdots, \zeta_n), \quad \zeta_1 \ge \zeta_2 \ge \cdots \ge \zeta_n \quad (5) \\ T_kM_kT_k^T &= T_k^{-T}N_kT_k^{-1} \\ &= diag(\eta_1, \eta_2, \cdots, \eta_r, \eta_{r+1}, \cdots, \eta_m), \\ &\quad \eta_1 \ge \eta_2 \ge \cdots \ge \eta_m \quad (6) \end{aligned}$$

with $\Sigma_{k1} = diag(\eta_1, \cdots, \eta_r)$, $\Sigma_{k2} = diag(\eta_{r+1}, \cdots, \eta_m)$. It is assumed that Σ_{k1} and Σ_{k2} have no diagonal entries in common and r is the order of the reduced order model to be found. Using the similarity transformation matrix T_k, the structurally balanced realization of $K(s)$ is obtained, and it is partitioned conformably with $diag(\Sigma_{k1}, \Sigma_{k2})$ as

$$K(s) = \left[\begin{array}{c|c} T_k A T_k^{-1} & T_k B \\ \hline C T_k^{-1} & D \end{array}\right] =: \left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right]$$
$$= \left[\begin{array}{cc|c} A_{b1} & A_{b2} & B_{b1} \\ A_{b3} & A_{b4} & B_{b2} \\ \hline C_{b1} & C_{b2} & D \end{array}\right]. \quad (7)$$

Theorem 1 [8]**:** *For the structurally balanced realization of $K(s)$, which is assumed to be RH_∞, the reduced order model by SBT is obtained as*

$$K_r(s) = \left[\begin{array}{c|c} A_{b1} & B_{b1} \\ \hline C_{b1} & D_b \end{array}\right]. \quad (8)$$

Let $T_{zwr}(s)$be a closed-loop transfer function by linear fractional transformation with $P(s)$ and$K_r(s)$, then

$$K_r(s) \in RH_\infty, \ \|T_{zw}(s) - T_{zwr}(s)\|_\infty \le 2tr(\Sigma_{k2}). \quad (9)$$

From theorem 1, it is to be seen that the stable reduced order controller with simple error bound is obtained when two block diagonal LMI solutions M and N exist. In the next section, the existence of the block diagonal LMI solutions for the weighted model reduction problem will be shown.

3. Frequency weighted reduction

Now we consider $K(s)$ as a transfer function model with input and output weighting functions, W_i and W_o, respectively, and assume that these functions are all in RH_∞. Then the frequency weighted model reduction problem is to find a reduced order model $K_r(s)$ such that

$$\|W_o(K - K_r)W_i\|_\infty < \gamma, \quad (10)$$

where γ is an *a priori* error bound, the realizations of W_i and W_o are given by

$$W_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] \text{ and } W_o(s) = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right], \quad (11)$$

respectively, and it is desired that the error bound is easy to find. From (1) in section 2, if we take $P_{11}(s) = 0$, $P_{12}(s) = W_o(s)$, $P_{21}(s) = W_i(s)$, $P_{22}(s) = 0$, the following state space realization is obtained.

$$P(s) = \begin{bmatrix} 0 & W_o(s) \\ W_i(s) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_o & 0 & 0 & B_o \\ 0 & A_i & B_i & 0 \\ \hline C_o & 0 & 0 & D_o \\ 0 & C_i & D_i & 0 \end{array}\right]. \quad (12)$$

Therefore, the state space realization of the closed-loop system $T_{zw}(s)$ with $P(s)$ and $K(s)$ is represented by the following formula:

$$T_{zw}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{ccc|c} A_o & B_oDC_i & B_oC & B_oDD_i \\ 0 & A_i & 0 & B_i \\ 0 & BC_i & A & BD_i \\ \hline C_o & D_oDC_i & D_oC & D_oDD_i \end{array}\right]. \quad (13)$$

The stable reduced order model with simple error bound is obtained by theorem 1 when two block diagonal LMI solutions M and N in (3) and (4) exist. Although it is very easy to get a solution by this method and the method gives a simple error bound without the computation of infinity-norm, the existence of the two block diagonal LMI solutions have not been verified. The following Lemma 1 is introduced as a preliminary step to prove the existence of the LMI solutions.

Lemma 1[9]**:** *For the symmetric matrix $L = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix}$, we have*

i) $L_{11} < 0$, $L_{22} - L_{12}^T L_{11}^{-1} L_{12} \le 0 \Rightarrow L \le 0$

ii) $L_{22} < 0$, $L_{11} - L_{12} L_{22}^{-1} L_{12}^T \le 0 \Rightarrow L \le 0$.

Theorem 2: *Consider the single-sided and the double-sided frequency weighted model reduction problems with a state space realization, (A_w, B_w, C_w, D_w). There exist nonnegative block diagonal solutions M and N such that*

$$A_w M + M A_w^T + B_w B_w^T \le 0, \quad (14)$$
$$A_w^T N + N A_w + C_w^T C_w \le 0, \quad (15)$$

where the structures of (A_w, B_w, C_w, D_w), M, and $N$are defined as follows:

i) For the right-sided weighting($\|(K - K_r)W_i\|_\infty$

$$\left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right] = \left[\begin{array}{cc|c} A_i & 0 & B_i \\ BC_i & A & BD_i \\ \hline DC_i & C & DD_i \end{array}\right], \quad (16)$$

$$M = diag(M_i, M_k), \ N = diag(N_i, N_k). \quad (17)$$

ii) For the left-sided weighting($\|W_o(K - K_r)\|_\infty$

$$\left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right] = \left[\begin{array}{cc|c} A_o & B_oC & B_oD \\ 0 & A & B \\ \hline C_o & D_oC & D_oD \end{array}\right], \quad (18)$$

$$M = diag(M_o, M_k), \ N = diag(N_o, N_k). \quad (19)$$

iii) For the double-sided weighting($\|W_o(K - K_r)W_i\|_\infty$)

$$\left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \end{array}\right] = \left[\begin{array}{ccc|c} A_o & B_oDC_i & B_oC & B_oDD_i \\ 0 & A_i & 0 & B_i \\ 0 & BC_i & A & BD_i \\ \hline C_o & D_oDC_i & D_oC & D_oDD_i \end{array}\right], \quad (20)$$

$$M = diag(M_{11}, M_{22}, M_k), \ N = diag(N_{11}, N_{22}, N_k). \quad (21)$$

Proof: i) Substituting (16) and (17) to (14), we have

$$\begin{bmatrix} A_iM_i + M_iA_i^T + B_iB_i^T & M_iC_i^TB^T + B_iD_i^TB^T \\ BC_iM_i + BD_iB_i^T & AM_k + M_kA^T + BD_iD_i^TB^T \end{bmatrix} \le 0. \quad (22)$$

Let L be the left-sided matrix in (22), L_{ij} being the i th row and jth column submatrix. Since A_i is a stable matrix, there exists nonnegative M_i satisfying $L_{11} < 0$. And given that $R = -L_{11}(> 0)$, then R^{-1} exists. From the second condition in i) of Lemma 1 we require

$$L_{22} + L_{12}^T R^{-1} L_{12} \le 0, \quad (23)$$

Furthermore, since A is a stable matrix, there exists M_k satisfying (23). Thus it is shown that there exists a block diagonal nonnegative solution M satisfying (14). Now substituting (16) and (17) to (15), we get the following LMI:

$$\begin{bmatrix} A_i^T N_i + N_i A_i + C_i^T D^T D C_i & C_i^T B^T N_k + C_i^T D^T C \\ N_k B C_i + C^T D C_i & A^T N_k + N_k A + C^T C \end{bmatrix} \leq 0. \tag{24}$$

Similar to (22), let L be the left-sided matrix in (24), L_{ij} being the ith row and jth column submatrix. Since A is a stable matrix, there exists nonnegative N_k satisfying $L_{22} < 0$. And given that $S = -L_{22} (> 0)$, then we require

$$L_{11} + L_{12} S^{-1} L_{12}^T \leq 0, \tag{25}$$

from the second condition in ii) of Lemma 1. Since A_i is a stable matrix, there exists nonnegative N_i satisfying (25). Therefore it is proved that there exists a nonnegative block diagonal solution N satisfying (15).

ii) Substituting (18) and (19) to (14) and (15) respectively, we get the following two LMIs:

$$\begin{bmatrix} A_o M_o + M_o A_o^T + B_o D D^T B_o^T & B_o C M_k + B_o D B^T \\ M_k C^T B_o^T + B D^T B_o^T & A M_k + M_k A^T + B B^T \end{bmatrix} \leq 0, \tag{26}$$

$$\begin{bmatrix} A_o^T N_o + N_o A_o + C_o^T C_o & N_o B_o C + C_o^T D_o C \\ C^T B_o^T N_o + C^T D_o^T C_o & A^T N_k + N_k A + C^T D_o^T D_o C \end{bmatrix} \leq 0. \tag{27}$$

The existence of nonnegative block diagonal solutions M and N satisfying (14) and (15) respectively could be easily proved just as in the proof of part i). So we will skip the proof.

iii) Substituting (20) and (21) to (14), we get the following matrix L.

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}^T & L_{22} & L_{23} \\ L_{13}^T & L_{23}^T & L_{33} \end{bmatrix} = \begin{bmatrix} A_o M_{11} + M_{11} A_o^T + B_o D D_i D_i^T D^T B_o^T & B_o D C_i M_{22} + B_o D D_i B_i^T & B_o C M_k + B_o D D_i D_i^T B^T \\ * & A_i M_{22} + M_{22} A_i^T + B_i B_i^T & M_{22} C_i^T B^T + B_i D_i^T B^T \\ * & * & A M_k + M_k A^T + B D_i D_i^T B^T \end{bmatrix} \tag{28}$$

Now there exists $M_{22} = M_{22}^T \geq 0$ such that $L_{22} < 0$ since A_i is stable. Then L_{22} and L_{23} are known. Next, there exists $M_k = M_k^T \geq 0$ such that $L_{33} - L_{23}^T {L_{22}}^{-1} L_{23} < 0$ since A is stable. From the well-known schur complement, this proves that $\begin{bmatrix} L_{22} & L_{23} \\ L_{23}^T & L_{33} \end{bmatrix} < 0$. Also, L_{12} and L_{13} are known. Finally, there exists $M_{11} = M_{11}^T \geq 0$ such that

$$L_{11} - \begin{bmatrix} L_{12} & L_{13} \end{bmatrix} \begin{bmatrix} L_{22} & L_{23} \\ L_{23}^T & L_{33} \end{bmatrix}^{-1} \begin{bmatrix} L_{12}^T \\ L_{13}^T \end{bmatrix} \leq 0 \tag{29}$$

since A_o is stable. This proves that $L \leq 0$ from lemma 1. Now substituting (20) and (21) to (15), we get the following matrx L:

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12}^T & L_{22} & L_{23} \\ L_{13}^T & L_{23}^T & L_{33} \end{bmatrix} = \begin{bmatrix} A_o^T N_{11} + N_{11} A_o + C_o^T C_o & N_{11} B_o D C_i + C_o^T D_o D C_i & N_{11} B_o C + C_o^T D_o C \\ * & A_i^T N_{22} + N_{22} A_i + C_i^T D^T D_o^T D_o D C_i & C_i^T B^T N_k + C_i^T D^T D_o^T D_o C \\ * & * & A^T N_k + N_k A + C^T D_o^T D_o C \end{bmatrix}, \tag{30}$$

there exists $N_{11} = N_{11}^T \geq 0$ such that $L_{11} < 0$ since A_o is stable. Then L_{12} and L_{13} are known. Next, consider the following submatrix

$$\begin{bmatrix} L_{22} & L_{23} \\ L_{23}^T & L_{33} \end{bmatrix} = \begin{bmatrix} A_i^T & C_i^T B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} N_{22} & 0 \\ 0 & N_k \end{bmatrix} + \begin{bmatrix} N_{22} & 0 \\ 0 & N_k \end{bmatrix} \begin{bmatrix} A_i & 0 \\ B C_i & A \end{bmatrix} + \begin{bmatrix} C_i^T D^T D_o^T \\ C^T D_o^T \end{bmatrix} \begin{bmatrix} D_o D C_i & D_o C \end{bmatrix}, \tag{31}$$

then a proof similar to part i) shows that the positive semidefinite solution $diag(N_{22}, N_k)$ exists such that

$$\begin{bmatrix} L_{22} & L_{23} \\ L_{23}^T & L_{33} \end{bmatrix} - \begin{bmatrix} L_{12}^T \\ L_{13}^T \end{bmatrix} {L_{11}}^{-1} \begin{bmatrix} L_{12} & L_{13} \end{bmatrix} \leq 0 \tag{32}$$

since $\begin{bmatrix} A_i & 0 \\ B C_i & A \end{bmatrix}$ is stable. This proves that $L \leq 0$ from lemma 1. From these results we conclude that there exist the block diagonal solutions M and N satisfying (14) and (15) respectively. □

If we assume that $M = diag(\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, M_k)$ and $N = diag(\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix}, N_k)$ with $M_{12} \neq 0$ and $N_{12} \neq 0$, the existence of positive semidefinite solution M and N is not guaranteed, even if it is more reasonable assumption.

Corollary 1: *If $K(s)$ is strictly proper($D_K = 0$), then $M = diag(\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}, M_k)$ and $N = diag(\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix}, N_k)$ always exist such that two LMIs (14) and (15) are satisfied with (A_w, B_w, C_w, D_w) of (20).*

Proof: Substituting (20) and M to (14) and comparing the left-sided matrix of the obtained inequality to the symmetric matrix L in lemma 1, we get the following submatrices L_{11}, L_{12}, and L_{22}.

$$\begin{aligned} L_{11} &= A_{wo} M_{11} + \begin{bmatrix} B_{wo} D_K C_{wi} & B_{wo} C_K \end{bmatrix} \begin{bmatrix} M_{12}^T \\ 0 \end{bmatrix} \\ &\quad + M_{11} A_{wo}^T + \begin{bmatrix} M_{12} & 0 \end{bmatrix} \begin{bmatrix} C_{wi}^T D_K^T B_{wo} \\ C_K^T B_{wo}^T \end{bmatrix} \\ &\quad + B_{wo} D_K D_{wi} D_{wi}^T D_K^T B_{wo}^T, \\ L_{12} &= A_{wo} \begin{bmatrix} M_{12} & 0 \end{bmatrix} + \begin{bmatrix} B_{wo} D_K C_{wi} & B_{wo} C_K \end{bmatrix} \\ &\quad \times \begin{bmatrix} M_{22} & 0 \\ 0 & M_k \end{bmatrix} + \begin{bmatrix} M_{12} & 0 \end{bmatrix} \begin{bmatrix} A_{wi}^T & C_{wi}^T B_K^T \\ 0 & A_K^T \end{bmatrix} \\ &\quad + B_{wo} D_K D_{wi} \begin{bmatrix} B_{wi}^T & D_{wi}^T B_K^T \end{bmatrix}, \\ L_{22} &= \begin{bmatrix} A_{wi} & 0 \\ B_K C_{wi} & A_K \end{bmatrix} \begin{bmatrix} M_{22} & 0 \\ 0 & M_k \end{bmatrix} \\ &\quad + \begin{bmatrix} M_{22} & 0 \\ 0 & M_k \end{bmatrix} \begin{bmatrix} A_{wi}^T & C_{wi}^T B_K^T \\ 0 & A_K^T \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{wi} \\ B_K D_{wi} \end{bmatrix} \begin{bmatrix} B_{wi}^T & D_{wi}^T B_K^T \end{bmatrix}. \end{aligned} \tag{33}$$

Since $\begin{bmatrix} A_{wi} & 0 \\ B_K C_{wi} & A_K \end{bmatrix}$ is a stable matrix, there exists $diag(M_{22}, M_k) \geq 0$ satisfying $L_{22} < 0$, if $K(s)$ is strictly proper($D_K = 0$), then $M_{11}(\geq 0)$ and M_{12} exist such that $L_{11} + L_{12}(-L_{22})^{-1}L_{12}^T \leq 0$. Now substituting (20) and N to (15) and comparing the left-sided matrix of the obtained inequality to the symmetric matrix L in lemma1, we get the following submatrices L_{11}, L_{12}, and L_{22}.

$$\begin{aligned}
L_{11} &= A_{wo}^T N_{11} + N_{11}A_{wo} + C_{wo}^T C_{wo}, \\
L_{12} &= A_{wo}^T \begin{bmatrix} N_{12} & 0 \end{bmatrix} + N_{11} \begin{bmatrix} B_{wo}D_K C_{wi} & B_{wo}C_K \end{bmatrix} \\
&\quad + \begin{bmatrix} N_{12} & 0 \end{bmatrix} \begin{bmatrix} A_{wi} & 0 \\ B_K C_{wi} & A_K \end{bmatrix} \\
&\quad + C_{wo}^T \begin{bmatrix} D_{wo}D_K C_{wi} & D_{wo}C_K \end{bmatrix}, \\
L_{22} &= \begin{bmatrix} C_{wi}^T D_K^T B_{wo}^T \\ C_K^T B_{wo}^T \end{bmatrix} \begin{bmatrix} N_{12} & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} A_{wi}^T & C_{wi}^T B_K^T \\ 0 & A_K^T \end{bmatrix} \begin{bmatrix} N_{22} & 0 \\ 0 & N_k \end{bmatrix} \\
&\quad + \begin{bmatrix} N_{12}^T \\ 0 \end{bmatrix} \begin{bmatrix} B_{wo}D_K C_{wi} & B_{wo}C_K \end{bmatrix} \\
&\quad + \begin{bmatrix} N_{22} & 0 \\ 0 & N_k \end{bmatrix} \begin{bmatrix} A_{wi} & 0 \\ B_K C_{wi} & A_K \end{bmatrix} \\
&\quad + \begin{bmatrix} C_{wi}^T D_K^T D_{wo}^T \\ C_K^T D_{wo}^T \end{bmatrix} \begin{bmatrix} D_{wo}D_K C_{wi} & D_{wo}C_K \end{bmatrix}, \quad (34)
\end{aligned}$$

similarly, since the positive semidefinite solution N_{11} exists such that $L_{11} < 0$, $diag(N_{22}, N_k)(\geq 0)$ and N_{12} exist such that $L_{22} + L_{12}^T(-L_{11})^{-1}L_{12} \leq 0$. And it means that we can find $M_{11}(\geq 0)$ such that $L_{11} + L_{12}(-L_{22})^{-1}L_{12}^T \leq 0$ for any M_{12}, and find $diag(N_{22}, N_k)(\geq 0)$ such that $L_{22} + L_{12}^T(-L_{11})^{-1}L_{12} \leq 0$ for any N_{12}.□

Note that the stability of reduced order model is guaranteed by the proposed method, and that the error bound is obtained as follows:

$$\|W_o(K - K_r)W_i\|_\infty \leq 2tr(\Sigma_{k2}), \quad (35)$$

where Σ_{k2} is the structurally balancing solution for the truncated model as defined in (6). This error bound is an *a priori* error bound, and it is very simple to get the error bound and it is not necessary to compute the infinity-norm of transfer functions unlike other methods proposed by Kim [6], Sreeram [5], and Wang [7]. Considering the error bound in (35), it is desirable to choose M and N such that $\Sigma_{i=r+1}^m \lambda_i^{1/2}(M_k N_k)$ is minimized. However, since the optimization involved is not convex, such solutions are hard to compute. Then we have the following procedure:

i) Find $M = diag(M_1, M_k)$ that minimizes $\Sigma_{i=1}^m \lambda_i(M_k) = tr(M_k)$ subject to (14). Similarly find $N = diag(N_1, N_k)$ that minimizes $\Sigma_{i=1}^m \lambda_i(N_k) = tr(N_k)$ subject to (15).

ii) Find a nonsingular T_k using any existing balancing algorithm so that (14) is satisfied.

iii) A reduced-order model K_r is obtained as (8) such that $K_r \in RH_\infty$ and the error bound (35) is satisfied.

4. Numerical example

To demonstrate the validity of proposed method, we present two examples which have been used by Sreeram [5] and Wang [7], and the actual error and error bound are compared with other existing methods.

Example 1: This example has been used by Sreeram [5]. For this example, reduced order models of order1-2 are obtained using: four different methods: 1) Enns' method; 2) Lin and Chiu's method; 3) Wang's method; 4) the proposed method. The error bounds are computed using Kim *et al.*, Sreeram *et al.*, Wang *et al.*, and the proposed method. Consider the third-order system

$$\frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}$$

with the following input and output weights:

$$\frac{1}{s + 5.8}, \text{ and } \frac{1}{s + 4}$$

The following table gives a comparison of errors and errors bounds for the models obtained by the proposed method and other existing methods.

Table 1. The errors and error bounds with input and output weightings; error(error bound)

order	Enns' method	Lin & Chiu's method	Wang's method	proposed method
1	0.0883 (0.3828)	0.0885 (0.2257)	0.08835 (0.2709)	0.0597 (0.0696)
2	0.0067 (0.0070)	0.0074 (0.0087)	0.00709 (0.0340)	0.00704 (0.00704)

Using table 1, show that it is possible to get lower errors and error bounds with the proposed technique than with the other methods, except for 2nd-order case with the Enns' method. The error bound in the proposed method is less conservative than other methods.

Example 2: This example has been used by Wang[7]. Consider a fourth-order system given by

$$A = \begin{bmatrix} -0.6503 & -0.2734 & 0.0030 & -0.1815 \\ 0.2883 & -1.0171 & 0.0102 & -1.2651 \\ 0.0377 & 0.1087 & -0.0011 & -3.2129 \\ 0.8699 & -4.6643 & 16.1671 & -18.3349 \end{bmatrix}$$

$$B = \begin{bmatrix} 3.3317 & 3.2155 \\ -1.9209 & -0.0978 \\ -4.5402 & 2.6599 \\ -17.4882 & 6.0988 \end{bmatrix}$$

$$C = \begin{bmatrix} 31.5142 & 6.4374 & -0.0750 & 4.3834 \end{bmatrix}$$

with the following input weight:

$$A_i = \begin{bmatrix} -8 & 0 \\ 1 & -3 \end{bmatrix}, B_i = \begin{bmatrix} 3 \\ 10 \end{bmatrix}, C_i = \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}, D_i = \begin{bmatrix} 1 \\ 2.17 \end{bmatrix}$$

The results are represented in the table 2. For this example, Enns' method gives very large errors, and the error bound is very conservative. The actual error by the proposed method may be smaller or larger than other methods, but the error bound is less conservative with the proposed method than with other methods.

There are two values on the proposed method for each order. The 2nd row value is obtained from M and N such that minimize the trace of a submatrix of M_k and N_k respectively.

Table 2. The errors and error bounds with input weightings

order	Enns' method	error bound	Wang's method	error bound	proposed method	error bound
1	39.5692	5479.8	38.4689	96.5676	45.5450	59.0401
					43.4354	59.6946
2	1.3787	5427.7	3.4582	19.9932	4.8944	15.5846
					3.2617	14.5691
3	594.7	5414.5	6.2949	8.4325	5.3514	5.4070
					5.6634	5.7437

It means that, if the actual error becomes smaller, the minimization index for the LMI solutions M_k and N_k can be different from the procedure in section 2.

5. Conclusions

In this paper, a simple frequency weighted model reduction technique using structurally balanced truncation has been proposed, and we have proved that two block diagonal LMI solutions always exist, and obtained a reduced order model with guaranteed stability and *a priori* error bound. One of advantages of the proposed method is that the error bound is easily obtained from the LMI solutions without infinity-norm computation of transfer functions. Two illustrative examples have been given to demonstrate the validity of the proposed method.

References

[1] B. C. Moore, "Principal component analysis in linear systems: Controllability, Observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, pp. 17-32, 1981.

[2] K. Glover, "All optimal Hankel-norm approximation of linear multivariable systems and their L_∞ error bounds," *International Journal of Control*, vol. 39, pp. 1115-1193, 1984.

[3] D. F. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," *Proceedings of 23rd IEEE Conference on Decision and Control, Las Vegas, USA*, pp. 127-132, 1984.

[4] C. A. Lin, and T. Y. Chiu, "Model reduction via frequency weighted balanced realization," *Control Theory and Advanced Technology*, vol. 8, pp. 341-451, 1992.

[5] V. Sreeram, B. D. O. Anderson, and A. G. Madievski, "frequency weighted balanced reduction technique," *Proceedings of American Control Conference, Washington, USA,* pp. 4004-4009, 1995.

[6] S. W. Kim, B. D. O. Anderson, and A. G. Madievski, "Error bound for transfer function order reduction using frequency weighted balanced truncation," *Systems and Control Letters*, vol. 24, pp. 183-192, 1995.

[7] G. Wang, V. Sreeram, and W. Q. Liu, "A new frequency-weighted balanced truncation method and an error bound," *IEEE Transactions on Automatic Control*, vol. 44, pp. 1734-1737, 1999.

[8] K. Zhou, C. D'Souza, and J. R. Cloutier, "Structurally balanced controller order reduction with guaranteed closed loop performance," *Systems and Control Letters*, vol. 24, pp. 235-242, 1995.

[9] S. Boyd, L. E. Ghaoui, E. Feron, and G. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics., 1994.