Design of a dynamic output feedback law for replacing the output derivatives

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Abstract: This paper provides a design method for a dynamic output feedback controller which stabilizes a class of linear time invariant systems. We suppose all the states of the given system is not measurable and only the outputs are used to stabilize the system. The systems considered cannot be stabilized by a static output feedback only. In the scheme we first assume that the given system can be stabilized by a state feedback composed of its output, velocity of the output and its higher order derivative terms. Instead of using the derivatives of the output, however, a dynamic system is constructed systematically which replaces the role of the derivative terms. Then, a high-gain output feedback stabilizes the composite system together with the newly constructed system. The performance of the proposed control law is illustrated in the comparative simulation studies of a numerical example with an observer-based control law.

Keywords: Linear System, Output Derivatives, Dynamic Ouput Feedback, Observer, Passivity

1. Introduction

The design of stabilizing controllers for linear time-invariant (LTI) systems has received considerable attention in the last several decades [1, 2]. In this paper, we consider the stabilization problem of a system represented by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where x is the state in $\mathbb{R}^n$; u the input in $\mathbb{R}^m$; y the measurable output in $\mathbb{R}^p$.

We suppose the system (1) cannot be stabilized by a static output feedback only. When the measurable states are not sufficient to design a stabilizing control law, a dynamic output feedback scheme with an additional dynamic system e.g. state observer is designed so that the augmented closed loop system is stable. In a recent result [3] a passivity-based dynamic output feedback control law has been proposed for inherently non-passive LTI systems by using a parallel feedforward compensator. If a given system can be stabilized by the PD (proportional-derivative) control law, it has been shown that the output derivative terms in the PD law can be replaced with the compensator which has the same dimension with the system's input. Hence, the systems can be stabilized without using the first output derivative terms.

We remind the readers of several notions which have been used in the passivity-based controller design methods. A system is minimum phase if its zero dynamics (ZD) subsystem is (asymptotically) stable [4]. The system (1) has relative degree one if the matrix CB is nonsingular (when the system is square). A square LTI system can be rendered strictly passive by a static output feedback if and only if it is minimum phase and has relative degree one [3, 5]; this kind of system is called almost strictly passive. More specifically, if the system (1) is minimum phase and $CB = I$, then the system can be transformed into the following normal form [4]:

$$\begin{aligned} \dot{\zeta} &= A_{\zeta a}\zeta + A_{\zeta b}y \\ \dot{y} &= A_{ya}\zeta + A_{yb}y + u \end{aligned} \tag{2}$$

where $A_{\zeta a}$ is Hurwitz (i.e. stable). It can be easily shown that there exists a matrix gain Ψ such that the control input $u = \Psi y$ stabilizes the system (2) (or, equivalently, (1)).

We present in this paper a new controller design scheme for broader class of systems from [3], in which the additional dynamic system has the same dimension with the system's input. We consider the systems which require higher order output derivatives for a stabilizing control law rather than the first order term. Hence, the results of the paper is some extensions of the previous result [3] to the construction of a higher order additional system which stabilizes broader class of systems than what was considered in [3].

The only assumption in this paper is the following:

Assumption 1. *Let us define*

$$\bar{K}_k := \begin{bmatrix} K_0 & K_1 & \cdots & K_k \end{bmatrix} \quad \text{and} \quad H_k := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix}.$$

For the system (1), *there exist an integer r $(1 \le r)$ and some matrices $K = \bar{K}_r$ and P of appropriate dimensions satisfying*

$$PA_r + A_r^T P < 0 \tag{3}$$

where $A_r = A + BKH_r$ and $P > 0$.

Remark 1. It is presupposed in this assumption that $r \geq 1$. When the condition (3) holds with $r = 0$, the system (1) can be stabilized by a static output feedback without using additional dynamics. If the system (1) is stabilizable and observable, then the condition (3) becomes easy to hold as r increases. Hence, each r characterizes its own class of systems.

In the next section, we present a dynamic controller for (1) only under Assumption 1, followed by the recursive algorithm to design the gains of the proposed controller in a systematic manner. Section 3 illustrates a design example with a simulation result. Conclusions are found in Section 4.

2. Main Results

For the system (1) that satisfies Assumption 1, we propose a dynamic output feedback controller of the form:

$$\begin{aligned}\dot{\lambda} &= \Psi_a Cx + \Psi_b \lambda \\ u &= \Phi_a Cx + \Phi_b \lambda\end{aligned} \tag{4}$$

where $\lambda \in \mathbb{R}^{rp}$ is the internal states of additional dynamics (thus, we know their values). Then, the stabilization problem is solved if we design $\Phi = [\Phi_a, \Phi_b]$ and $\Psi = [\Psi_a, \Psi_b]$ matrices so that the states $x(t) \to 0$ and $\lambda(t) \to 0$ as $t \to \infty$ for the following closed loop system:

$$\begin{aligned}\dot{x} &= Ax + B\Phi_a Cx + B\Phi_b \lambda \\ \dot{\lambda} &= \Psi_a Cx + \Psi_b \lambda.\end{aligned} \tag{5}$$

In the subsequent part of the paper, we will show the design of the matrices Φ and Ψ by a recursive algorithm. Therefore, the main contribution of the paper is summarized as

Theorem 2. *For the system* (1) *satisfying Assumption 1, there exists a dynamic output feedback stabilizing controller* (4) *with additional λ-dynamics of order* $(r \times p)$.

The idea of the construction of (4) is to assume, in the beginning, that $H_r x$ is available for measurement on the basis of Assumption 1. As the first step of the construction, we change our virtual assumption so that $H_{r-1}x$ is available for measurement but $CA^r x$ is not. Thus, the designed control law when $H_r x$ is all measurable is not implementable because it depends on the signal $CA^r x$. We extract the signal $CA^r x$ from the control law and design additional dynamics with which the use of $CA^r x$ is eliminated. In the next step, we proceed by assuming that $H_{r-2}x$ is measurable but $CA^{r-1}x$ is not. The recursion goes to the end if we get a dynamic controller that requires only the measurement of $H_0 x = Cx$ but not others.

The recursion begins by the following initial step.

2.1. Initial Step

When the $H_r x$ is measurable, under Assumption 1, the closed loop system S_r with the following control law is stable:

$$S_r : \begin{cases} u &= KH_r x = \bar{K}_{r-1}H_{r-1}x + K_r(CA^r x) \\ \dot{x} &= A_r x = (A + BKH_r)x \\ &= Ax + B\bar{K}_{r-1}H_{r-1}x + BK_r(CA^r x) \end{cases} \tag{6}$$

where K is given in Assumption 1.

As the initial step for designing the λ-dynamics in (4) we assume that $H_{r-1}x$ is available for measurement but $CA^r x$ is not. Then, by introducing v, we decompose the system S_r into the term including $CA^k x$ and the rest (as follows):

$$u = \bar{K}_{r-1}H_{r-1}x + K_r v \tag{7a}$$

$$\dot{x} = Ax + B\bar{K}_{r-1}H_{r-1}x + BK_r v. \tag{7b}$$

If the following dynamic system is appended to (7b)

$$\dot{\lambda} = -CA^{r-1}B\bar{K}_{r-1}H_{r-1}x - (I + CA^{r-1}BK_r)v \tag{8a}$$

$$\bar{y} = CA^{r-1}x + \lambda, \tag{8b}$$

then the augmented system (7b)–(8a) is stabilized by $v = \Psi_r \bar{y}$ where Ψ_r is chosen so that

$$A_{r-1} := \begin{bmatrix} A_r & -A_r BK_r \\ CA^r & -CA^r BK_r - \Psi_r \end{bmatrix} \text{ is Hurwitz.} \tag{9}$$

Proof of Initial Step.
First of all, note that

$$\begin{aligned}\dot{\bar{y}} &= CA^{r-1}\dot{x} + \dot{\lambda} \\ &= CA^{r-1}(Ax + B\bar{K}_{r-1}H_{r-1}x + BK_r v) \\ &\quad - (CA^{r-1}B\bar{K}_{r-1}H_{r-1}x + CA^{r-1}BK_r v + v) \\ &= CA^r x - v.\end{aligned}$$

We now define

$$\xi = x + BK_r \bar{y} \tag{10}$$

and change coordinates $[x^T \;\; \lambda^T]^T$ into $[\xi^T \;\; \bar{y}^T]^T$. Then

$$\begin{aligned}\dot{\xi} &= A_r \xi - A_r BK_r \bar{y} \\ \dot{\bar{y}} &= CA^r \xi - CA^r BK_r \bar{y} - v.\end{aligned} \tag{11}$$

Since the matrix A_r is Hurwitz, the system (11) can be stabilized by $v = \Psi_r \bar{y}$ with an appropriate gain Ψ_r. For example, $\Psi_r = \psi_r I$ with sufficiently large $\psi_r > 0$.

Consequently, we obtain the closed loop system S_{r-1} as follows:

$$S_{r-1} : \begin{cases} u &= \bar{K}_{r-1}H_{r-1}x + K_r\Psi_r(CA^{r-1}x + \lambda) \\ &= \left(\bar{K}_{r-1} + [0 \;\; K_r\Psi_r]\right)H_{r-1}x + K_r\Psi_r\lambda \\ \dot{x} &= Ax + B\left(\bar{K}_{r-1} + [0 \;\; K_r\Psi_r]\right)H_{r-1}x \\ &\quad + BK_r\Psi_r\lambda \\ \dot{\lambda} &= -\left(CA^{r-1}B\bar{K}_{r-1} + [\,0 \;\; D_\lambda\,]\right)H_{r-1}x \\ &\quad - D_\lambda \lambda \end{cases} \tag{12}$$

where $D_\lambda = (I + CA^{r-1}BK_r)\Psi_r$. The above system (12) is stable because its system matrix is similar to A_{r-1} of (9). □

2.2. Recursive Design

Suppose that a system S_k (k is an index between 0 and r and the recursion begins when $k = r$ and ends with $k = 0$) given by

$$S_k : \begin{cases} u = K_a H_k x + K_b \lambda \\ \quad = K_{a1}H_{k-1}x + K_b\lambda + K_{a2}(CA^k x) \\ \dot{x} = Ax + BK_a H_k x + BK_b \lambda \\ \quad = Ax + D_{1a}H_{k-1}x + D_{12}\lambda + D_{1b}(CA^k x) \\ \dot{\lambda} = D_{21}H_k x + D_{22}\lambda \\ \quad = D_{2a}H_{k-1}x + D_{22}\lambda + D_{2b}(CA^k x) \end{cases} \tag{13}$$

where $\lambda \in \mathbb{R}^{p(r-k)}$; $BK_a = [D_{1a} \;\; D_{1b}]$ and $BK_b = D_{12}$. The matrices A and H_k (from A and C) are given in (1), and all D matrices have appropriate dimensions. Note that λ is null when $k = r$ (i.e. S_r), but increases its dimension as the recursion proceeds.

Let $z = [x^T \;\; \lambda^T]^T$. If v is taken as

$$v = CA^k x, \tag{14}$$

the system S_k will be concisely denoted by

$$\dot{z} = A_k z = Fz + D_b v \tag{15}$$

where

$$A_k = \begin{bmatrix} A + BK_a H_k & BK_b \\ D_{21} H_k & D_{22} \end{bmatrix};$$

$F = \begin{bmatrix} A + D_{1a} H_{k-1} & D_{12} \\ D_{2a} H_{k-1} & D_{22} \end{bmatrix}$ and $D_b = \begin{bmatrix} D_{1b} \\ D_{2b} \end{bmatrix}$. Note that $A_k = F + D_b [CA^k \;\; 0]$.

Since $CA^k x$ is not available for measurement (when $k \geq 1$), we now assume that $H_{k-1} x$ is available for measurement but $CA^k x$ is not. Then, the following theorem shows that, by attaching additional dynamics, we can design an alternative v that does not depend on the unmeasurable quantity $CA^k x$.

Theorem 3. *Suppose that the system* (15) *is stable when* v *is taken as* (14)*, i.e. the matrix* A_k *is Hurwitz. If the following dynamic system is appended to* (15)

$$\begin{aligned} \dot{\eta} &= -CA^{k-1} D_{1a} H_{k-1} x - CA^{k-1} D_{12} \lambda \\ &\quad - (I + CA^{k-1} D_{1b}) v && \text{(16a)} \\ \bar{y} &= CA^{k-1} x + \eta, && \text{(16b)} \end{aligned}$$

then the augmented system (15)–(16) *is stabilized by redesigning*

$$v = \Psi_k \bar{y}, \tag{17}$$

in which $\bar{y}$ *is measurable if* $H_{k-1} x$ *is assumed to be measurable. The matrix gain* Ψ_k *is chosen so that*

$$A_{k-1} := \begin{bmatrix} A_k & -A_k D_b \\ \begin{bmatrix} CA^k & 0 \end{bmatrix} & -CA^k D_{1b} - \Psi_k \end{bmatrix} \tag{18}$$

is Hurwitz.

Remark 4. Under Assumption 1 (and from the initial step) the system S_k is stable when $k = r$ (and $k = r - 1$). For the rest of the cases the stability will be justified as the recursion proceeds. Note that the matrix (18) always can be Hurwitz e.g. when $\Psi_k = \psi_k I$ with sufficiently large $\psi_k > 0$.

Proof of Theorem 3

From (13)–(14)–(16)–(17) the proposed control law is given by

$$u = K_{a1} H_{k-1} x + K_b \lambda + K_{a2} \Psi_k (CA^{k-1} x + \eta). \tag{19}$$

Hence, the closed-loop system (15)–(16)–(19) can be written as a single system (20) in the next page.

As in the proof of Initial Step, we change coordinates with the following transformation matrix T:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{y} \end{pmatrix} = T \begin{pmatrix} x \\ \lambda \\ \eta \end{pmatrix} = \begin{bmatrix} I + D_{1b} CA^{k-1} & 0 & D_{1b} \\ D_{2b} CA^{k-1} & I & D_{2b} \\ CA^{k-1} & 0 & I \end{bmatrix} \begin{pmatrix} x \\ \lambda \\ \eta \end{pmatrix}. \tag{21}$$

Then we obtain

$$\begin{aligned} \dot{\xi} &= F\xi + D_b CA^k \xi_1 - (FD_b + D_b CA^k D_{1b}) \bar{y} \\ &= A_k \xi - A_k D_b \bar{y} \\ \dot{\bar{y}} &= [CA^k \;\; 0] \xi - CA^k D_{1b} \bar{y} - v. \end{aligned} \tag{22}$$

Since the matrix A_k is Hurwitz, there exists $v = \Psi_k \bar{y}$ that stabilizes the system (22). With this Ψ_k the matrix A_{k-1} of (18) is Hurwitz. For the system (20), it follows that

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{\eta} \end{bmatrix} = \tilde{A}_{k-1} \begin{bmatrix} x \\ \lambda \\ \eta \end{bmatrix}$$

where $\tilde{A}_{k-1} = T^{-1} A_{k-1} T$. Therefore, the stability of (20) has been proved. □

Finally the recursion procedure is quite obvious. By the initial step, the system S_{r-1} is stable and the proof of Theorem 3 presents the system S_{r-2} by the equation (20). Indeed the new D_{ij} matrices are identified by redefining $[\lambda^T, \eta^T]^T$ as the new λ and by extracting $CA^{r-2} x$ term. Then the system S_{r-2} is also stable, which enables to apply the recursion to the system S_{r-2} and the system S_{r-3} is obtained. This recursion will end with S_1, because Theorem 3 will yield an implementable control system (i.e., the system S_0). As a result, the system (20) will be the same as (5), and all matrices Φ and Ψ are derived straightforwardly.

3. Design Example

We illustrate the proposed design method with a simple numerical example:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned} \tag{23}$$

where the constant a represents an uncertainty and is assumed to be 1 in the design procedure.

The system (23) satisfies Assumption 1 with $r = 2$. In fact, with the following control law

$$u = KH_2 x = \begin{bmatrix} -50 & -40 & -11 \end{bmatrix} H_2 x, \tag{24}$$

the eigenvalues of the matrix $A_2 = A + BKH_2$ are given by $\{-5, -3 \pm j\}$. Hence, the closed loop system (23)–(24) is stable and we obtain $\bar{K}_1 = \begin{bmatrix} -50 & -40 \end{bmatrix}$ and $K_2 = -11$. In order to replace the $CA^2 x$-term in $H_2 x$, at the initial step, the matrix A_1 in (9) is considered for (23). The matrix is obtained by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 11 \\ -50 & -40 & -11 & -121 \\ 0 & 0 & 1 & -19 \end{bmatrix}$$

when the gain $\Psi_2 = 30$. Indeed the eigenvalues of A_1 have been chosen by $\{-12.83 \pm j9.38, -2.17 \pm j1.10\}$.

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A + D_{1a}H_{k-1} + D_{1b}\Psi_k \cdot CA^{k-1} & D_{12} & D_{1b}\Psi_k \\ D_{2a}H_{k-1} + D_{2b}\Psi_k \cdot CA^{k-1} & D_{22} & D_{2b}\Psi_k \\ -CA^{k-1}D_{1a}H_{k-1} - (I + CA^{k-1}D_{1b})\Psi_k CA^{k-1} & -CA^{k-1}D_{12} & -(I + CA^{k-1}D_{1b})\Psi_k \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ \eta \end{bmatrix}. \tag{20}$$

However, since the CAx-term in $H_1 x$ is neither measurable, we proceed one step further by Theorem 3. From the previous step, the parameters of (13) is given by

$$\begin{gathered} K_a = [-50 \quad -370],\ K_b = -330\ ; \\ D_{1a} = [0 \ \ 0 \quad -50]^T,\ D_{1b} = [0 \ \ 0 \quad -370]^T\ ; \\ D_{12} = [0 \ \ 0 \quad -330]^T,\ D_{22} = -30. \end{gathered} \tag{25}$$

With these parameters the gain Ψ_1 is chosen such that the matrix A_0 in (18) is Hurwitz. When $\Psi_1 = 30$ the eigenvalues of the matrix is given by $\{-45.12, -5.22 \pm j12.21, -2.22 \pm j0.85\}$.
Finally, with the following additional dynamics:

$$\begin{cases} \dot{\lambda} = -900y - 30\lambda - 900\eta \\ \dot{\eta} = -30y - 30\eta, \end{cases} \tag{26}$$

the stabilizing control law (19) for (23) is obtained by

$$u = -11150y - 330\lambda - 11100\eta. \tag{27}$$

Comparative simulation studies are performed with the proposed control law and the control law using a state observer [2]. Though the additonal dynamics (26) and the state observer are constructed when $a = 1$ in (23), in order to test the robust property of the proposed control law against parameter uncertainties, the simulations are carried out when $a = 2$ and $a = 3$. When $a = 2$ both control laws stabilize the system. Figure 1 shows the simulation results when the maximum amplitude of the control laws are constrained to 50. That is, we have placed the saturation operator at the output of both controllers. On the other hand, if the parameter $a = 3$, unlike the case when $a = 2$ the closed loop system with the observer-based control is unstable (even without the control saturation), and the system states diverge while the proposed control still stabilizes the system. The simulation results are given in Fig. 1, which shows the robust property of the proposed control law. In the figure we can see that the system states by the observer-based control are bounded since the control value is saturated. All the initial conditions of the system are set to 1 while all the initial states of the additional dynamics and the observer are set to 0.

4. Conclusion

In this paper, we present a new recursive algorithm to design a dynamic output feedback control law which stabilizes linear time-invariant systems. By Assumption 1, the class of systems that admits the dynamic output feedback controller is much broader than that of [3], and the index r characterizes the class. The recursive design indicates the higher order dynamics is necessary when the index r increases. By some computer simulations the proposed control law may have some advantages to the observer-based control when the given systems have some parameter uncertainties. From the proposed recursion algorithm, it seems easy to develop an automated design package on a PC.

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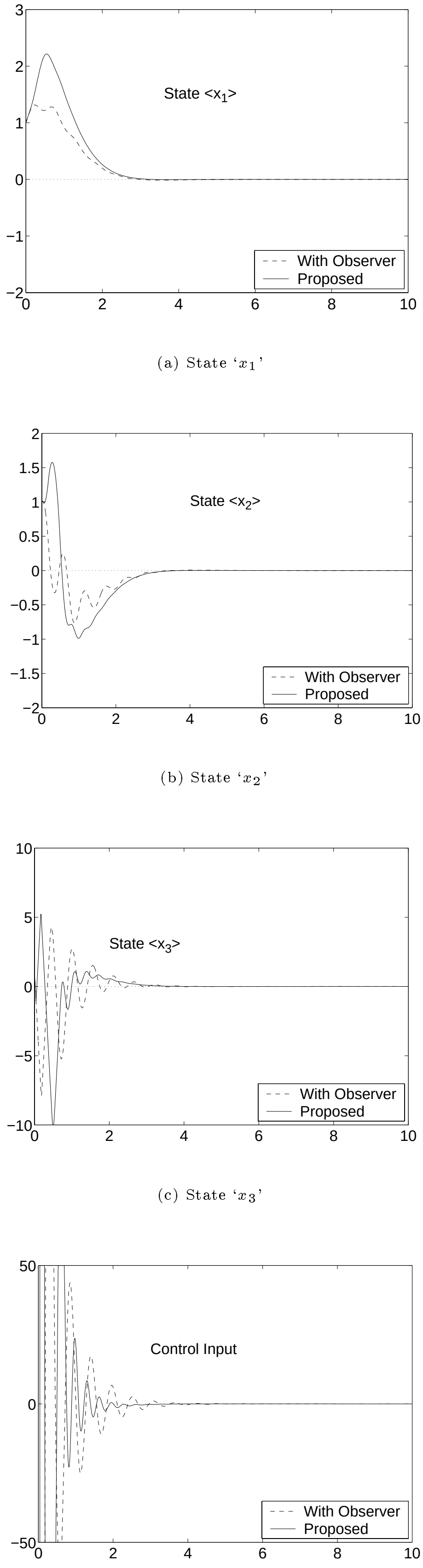


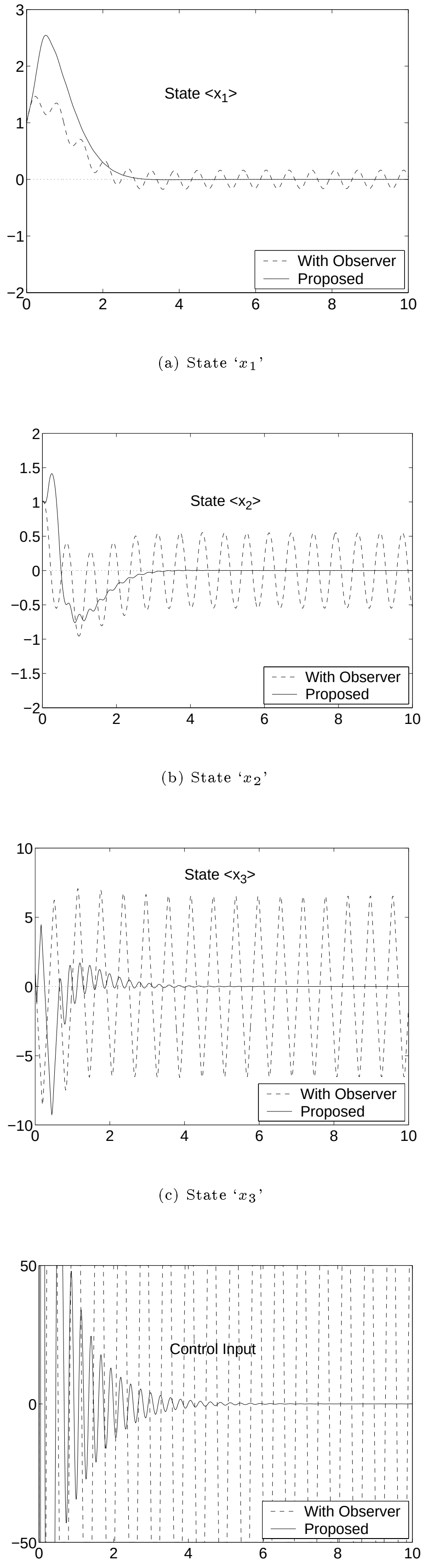
(a) State 'x_1'

(b) State 'x_2'

(c) State 'x_3'

(d) Control Input 'u'

Fig. 1. Simulation Results (proposed: solid).



(a) State 'x_1'

(b) State 'x_2'

(c) State 'x_3'

(d) Control Input 'u'

Fig. 2. Simulation Results II (proposed: solid).