New Robust $\mathcal{H}_\infty$ Performance Condition for Uncertain Discrete-Time Systems

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Abstract: In this paper, we establish a new robust $\mathcal{H}_\infty$ performance condition for uncertain discrete-time systems with convex polytopic uncertainties. We express the condition as a set of linear matrix inequalities (LMIs), which are used to check stability and $\mathcal{H}_\infty$ disturbance attenuation level by a parameter-dependent Lyapunov matrix. We show that the new condition provides less conservative result than the existing ones which use single Lyapunov matrix. We also show that the robust $\mathcal{H}_\infty$ state feedback design problem for such uncertain discrete-time systems can be easily dealt with using the approach. The key point in this paper is to propose a kind of decoupling between the Lyapunov matrix and the system matrices in the parameter-dependent matrix inequality by introducing one new matrix variable.

Keywords: Uncertain discrete-time system, robust $\mathcal{H}_\infty$ performance, parameter-dependent Lyapunov matrix, convex polytopic uncertainty, linear matrix inequality (LMI)

1. Introduction

Robust analysis and robust controller synthesis for uncertain systems has been a very hot field in control theory and has been investigated thoroughtly using various approach. For general introduction and results, refer to Refs. [1]-[6] and the references cited therein. However, there are still many open problems in this field; see Ref. [7] for details. Among the problems mentioned in [7], we focus our attention on the robust analysis problem attempting to answer the following question: "Given a set of uncertain systems with disturbance input and controlled output, and with the system matrices belonging to convex polytopic sets, decide for a given positive scalar γ whether each system of this set has robust $\mathcal{H}_\infty$ performance γ (i.e., is asymptotically stable and has $\mathcal{H}_\infty$ disturbance attenuation level less than γ) or not." Although this problem has been studied by using a quadratic Lyapunov function, the existing results in that context are quite conservative. There have been also a few attempts to consider parameter-dependent Lyapunov matrices so as to reduce such conservativeness [8, 9], but most of them are in the continuous-time domain. Our objective here is to seek less conservative solutions for this problem in the case of uncertain discrete-time systems. It is noted that the motivation of this paper and the above question originate from Ref. [10]. In that context, the robust stability analysis problem was dealt with, and a new condition involving parameter-dependent Lyapunov matrices was proposed by introducing a new matrix variable and decoupling the product between the Lyapunov matrix and the system matrices.

Encouraged by Ref. [10], we in this paper extend the result of [10] to the case of robust $\mathcal{H}_\infty$ performance analysis for uncertain discrete-time systems whose matrices are known to be in convex polytopic sets. It is known that the condition for such analysis problem is expressed as a matrix inequality with respect to the Lyapunov matrix. Instead of using parameter-independent Lyapunov matrices, we use a parameter-dependent Lyapunov matrix in order to obtain less conservative result. Then, the robust $\mathcal{H}_\infty$ performance analysis problem is reduced to solving the matrix inequality that include the products of the Lyapunov matrix and the system matrices. Since both the Lyapunov matrix and the system matrices are dependent on polytopic parameters, it is difficult to obtain LMI(s) from the matrix inequality. To overcome this difficulty, we modify the approach in [10] to deal with our matrix inequality. More precisely, we introduce one new matrix variable so that our matrix inequality will not involve the products of the Lyapunov matrix and the system matrices. Due to this decoupling, we can check the existence and compute the parameter-dependent Lyapunov matrix by dealing with a family of LMIs for the vertix matrices of the polytopes. More importantly, we claim that many synthesis problem can be dealt with by using our new condition. We will give an example of using the condition for robust $\mathcal{H}_\infty$ state feedback design of uncertain discrete-time systems.

This paper is organized as follows. In Section 2, we first describe the uncertain discrete-time system under consideration of analysis, where uncertainties of polytopic type are assumed to exist in all system matrices. Then, by introducing a new matrix variable, we propose a new condition in LMI form for the robust $\mathcal{H}_\infty$ performance analysis of the uncertain system on hand. In Section 3, we adopt the new condition to solve the robust $\mathcal{H}_\infty$ state feedback design problem. A numerical example is used to demonstrate the effectiveness of the result. Finally, Section 4 concludes the paper.

2. Robust $\mathcal{H}_\infty$ Performance Analysis

We consider the uncertain linear discrete-time system

$$\left\{ \begin{aligned} x[k+1] &= A(\alpha)x[k] + B(\zeta)w[k] \\ z[k] &= C(\rho)x[k] + D(\psi)w[k] \end{aligned} \right. \tag{1}$$

where $x[k] \in \Re^n$ is the state, $w[k] \in \Re^m$ is the disturbance input, $z[k] \in \Re^p$ is the controlled output. The matrices $A(\alpha), B(\zeta), C(\rho), D(\psi)$ are uncertain and assumed to belong to the convex polytopic sets defined as

$$\begin{aligned}
\mathcal{A} &\triangleq \left\{ A(\alpha): \;\; A(\alpha_) = \sum_{i=1}^{N_A} \alpha_i A_i \,, \; \sum_{i=1}^{N_A} \alpha_i = 1 \,, \; \alpha_i \geq 0 \right\} \\
\mathcal{B} &\triangleq \left\{ B(\zeta): \;\; B(\zeta) = \sum_{j=1}^{N_B} \zeta_j B_j \,, \; \sum_{j=1}^{N_B} \zeta_j = 1 \,, \; \zeta_j \geq 0 \right\} \\
\mathcal{C} &\triangleq \left\{ C(\rho): \;\; C(\rho) = \sum_{k=1}^{N_C} \rho_k C_k \,, \; \sum_{k=1}^{N_C} \rho_k = 1 \,, \; \rho_k \geq 0 \right\} \\
\mathcal{D} &\triangleq \left\{ D(\psi): \;\; D(\psi) = \sum_{l=1}^{N_D} \psi_l D_l \,, \; \sum_{l=1}^{N_D} \psi_l = 1 \,, \; \psi_l \geq 0 \right\}
\end{aligned} \tag{2}$$

where A_i, B_j, C_k, D_l are constant matrices denoting the extreme points of the polytopes.

Definition 1. The system (1) is said to have robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) if for any $A(\alpha)$, $B(\zeta)$, $C(\rho)$, $D(\psi)$ belonging to (2), all eigenvalues of $A(\alpha)$ have magnitude less than one and the $\mathcal{H}_\infty$ norm of the transfer function from w to z in (1) is less than γ. □

The following result is well known for robust $\mathcal{H}_\infty$ performance of discrete-time systems.

Lemma 1.[6] The system (1) has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) if and only if there exists a positive definite matrix $P(\alpha, \zeta, \rho, \psi)$ (abbreviated as $P_{\alpha\zeta\rho\psi}$) such that

$$\begin{bmatrix} -P_{\alpha\zeta\rho\psi} & P_{\alpha\zeta\rho\psi}A(\alpha) & P_{\alpha\zeta\rho\psi}B(\zeta) & 0 \\ A^T(\alpha)P_{\alpha\zeta\rho\psi} & -P_{\alpha\zeta\rho\psi} & 0 & C^T(\rho) \\ B^T(\zeta)P_{\alpha\zeta\rho\psi} & 0 & -\gamma I & D^T(\psi) \\ 0 & C(\rho) & D(\psi) & -\gamma I \end{bmatrix} < 0 \tag{3}$$

or equivalently,

$$\begin{bmatrix} P_{\alpha\zeta\rho\psi} - A^T(\alpha)P_{\alpha\zeta\rho\psi}A(\alpha) & -A^T(\alpha)P_{\alpha\zeta\rho\psi}B(\zeta) & C^T(\rho) \\ -B^T(\zeta)P_{\alpha\zeta\rho\psi}A(\alpha) & -B^T(\zeta)P_{\alpha\zeta\rho\psi}B(\zeta) + \gamma I & D^T(\psi) \\ C(\rho) & D(\psi) & \gamma I \end{bmatrix} > 0 \tag{4}$$

holds for all α, ζ, ρ and ψ in (2). □

According to Lemma 1, the robust $\mathcal{H}_\infty$ performance analysis problem for (1) is expressed as the problem of solving (3) or (4) with respect to $P_{\alpha\zeta\rho\psi}$. However, there is no general and systemic way to formally determine $P_{\alpha\zeta\rho\psi}$ as a function of the uncertain parameters $\alpha, \zeta, \rho, \psi$. Such kind of matrix $P(\cdot)$ is called a parameter-dependent Lyapunov matrix for robust $\mathcal{H}_\infty$ performance analysis. One way of addressing this problem is to look for a single Lyapunov matrix $P_{\alpha\zeta\rho\psi} = P$ which solves (3) or (4). However, this approach obviously leads to quite conservative results. For comparison, we state the existing result using single Lyapunov matrix in the following, which is quite straightforward from (3).

Lemma 2. The system (1) has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) if there exists a positive definite matrix P such that the LMI

$$\begin{bmatrix} -P & PA_i & PB_j & 0 \\ A_i^T P & -P & 0 & C_k^T \\ B_j^T P & 0 & -\gamma I & D_l^T \\ 0 & C_k & D_l & -\gamma I \end{bmatrix} < 0 \tag{5}$$

holds for all $i = 1, \cdots, N_A$; $j = 1, \cdots, N_B$; $k = 1, \cdots, N_C$ and $l = 1, \cdots, N_D$. □

Rather than a single (or common) Lyapunov matrix, we propose solving the matrix inequality (3) or (4) with respect to $P_{\alpha\zeta\rho\psi}$ directly in order to obtain better result than Lemma 2. It is easy to observe that the difficulty is in the existence of the products between $A(\alpha)$ $(B(\zeta))$ and $P_{\alpha\zeta\rho\psi}$ in (3) and (4), and such a difficulty does not disappear no matter what kind of equivalent transformation we make by using Schur Complement Method.

To overcome this difficulty, we modify the idea proposed in [10] to introduce a new matrix variable in (3) or (4) so that we can decouple the products between $A(\alpha)$ $(B(\zeta))$ and $P_{\alpha\zeta\rho\psi}$. To proceed, we first prove the following important result.

Theorem 1. The following statements are equivalent:

(i) There exists a positive definite matrix P such that

$$\begin{bmatrix} P - A^T P A & -A^T P B & C^T \\ -B^T P A & -B^T P B + \gamma I & D^T \\ C & D & \gamma I \end{bmatrix} > 0\,. \tag{6}$$

(ii) There exist a positive definite matrix P and a matrix G such that

$$\begin{bmatrix} P & A^T G^T & 0 & C^T \\ GA & G + G^T - P & GB & 0 \\ 0 & B^T G^T & \gamma I & D^T \\ C & 0 & D & \gamma I \end{bmatrix} > 0\,. \tag{7}$$

Proof. (i)→(ii): According to Lemma 1, (6) is equivalent to

$$\begin{bmatrix} -P & PA & PB & 0 \\ A^T P & -P & 0 & C^T \\ B^T P & 0 & -\gamma I & D^T \\ 0 & C & D & -\gamma I \end{bmatrix} < 0 \tag{8}$$

or

$$\begin{bmatrix} P & -PA & -PB & 0 \\ -A^T P & P & 0 & -C^T \\ -B^T P & 0 & \gamma I & -D^T \\ 0 & -C & -D & \gamma I \end{bmatrix} > 0\,. \tag{9}$$

Exchanging the first and the second rows and columns in the above inequality leads to

$$\begin{bmatrix} P & -A^T P & 0 & -C^T \\ -PA & P & -PB & 0 \\ 0 & -B^T P & \gamma I & -D^T \\ -C & 0 & -D & \gamma I \end{bmatrix} > 0\,. \tag{10}$$

Then, we pre- and post-multiply the above inequality by $\mathrm{diag}\{I, -I, I, -I\}$ to obtain

$$\begin{bmatrix} P & A^T P & 0 & C^T \\ PA & P & PB & 0 \\ 0 & B^T P & \gamma I & D^T \\ C & 0 & D & \gamma I \end{bmatrix} > 0\,. \tag{11}$$

This implies that (7) holds with $G = G^T = P$.

(ii)→(i): Pre- and post-multiplying (7) respectively by the row full rank matrix

$$\begin{bmatrix} I & -A^T & 0 & 0 \\ 0 & -B^T & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{12}$$

and its transpose, we obtain (6) immediately. This completes the proof. □

Based on Lemma 1 and Theorem 1, we obtain the following result.

Theorem 2. The system (1) has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) if there exist a positive definite matrix $P_{\alpha\zeta\rho\psi}$ and a matrix G such that

$$\begin{bmatrix} P_{\alpha\zeta\rho\psi} & A^T(\alpha) G^T & 0 & C^T(\rho) \\ GA(\alpha) & G + G^T - P_{\alpha\zeta\rho\psi} & GB(\zeta) & 0 \\ 0 & B^T(\zeta) G^T & \gamma I & D^T(\psi) \\ C(\rho) & 0 & D(\psi) & \gamma I \end{bmatrix} > 0 \tag{13}$$

holds for all α, ζ, ρ and ψ in (2). □

We notice that the products between $A(\alpha)$ $(B(\zeta))$ and $P_{\alpha\zeta\rho\psi}$ do not exist any more in Theorem 2. Hence, from Theorem 2 and the convexity of $A(\alpha)$, $B(\zeta)$, $C(\rho)$ and $D(\psi)$ in (2), we obtain the following new condition for robust $\mathcal{H}_\infty$ performance analysis of the system (1).

Theorem 3. The system (1) has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) if there exist positive definite matrices P_{ijkl} and a matrix G such that the LMI

$$\begin{bmatrix} P_{ijkl} & A_i^T G^T & 0 & C_k^T \\ GA_i & G + G^T - P_{ijkl} & GB_j & 0 \\ 0 & B_j^T G^T & \gamma I & D_l^T \\ C_k & 0 & D_l & \gamma I \end{bmatrix} > 0 \tag{14}$$

holds all $i = 1, \cdots, N_A$; $j = 1, \cdots, N_B$; $k = 1, \cdots, N_C$ and $l = 1, \cdots, N_D$. If this is the case, one parameter-dependent Lyapunov matrix is given by

$$P(\alpha, \zeta, \rho, \psi) = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \sum_{k=1}^{N_C} \sum_{l=1}^{N_D} \alpha_i \zeta_j \rho_k \psi_l P_{ijkl}\,. \tag{15}$$

Remark 1. Since Lemma 2 is a special case of Theorem 3 by setting $P_{ijkl} = G = P$ in (14), we claim that Theorem 3 is a better result than the existing result using single Lyapunov matrix. □

3. Robust State Feedback Design

In this section, we consider the state feedback design problem for the uncertain linear discrete-time system

$$\begin{cases} x[k+1] \;=\; A(\alpha)x[k] + B(\zeta)w[k] + B_2(\beta)u[k] \\ \quad\;\; z[k] \;=\; C(\rho)x[k] + D(\psi)w[k] \end{cases} \tag{16}$$

where, in addition to the notations we have described in Section 2, $u[k]$ is the control input and $B_2(\beta)$ is assumbed to be in the convex polytopic set

$$\mathcal{B}_2 \triangleq \left\{ B_2(\beta) : \; B_2(\beta) = \sum_{m=1}^{M} \beta_m B_{2m} \,, \right. \left. \sum_{m=1}^{M} \beta_m = 1 \,,\; \beta_m \geq 0 \right\} . \tag{17}$$

Then, the control problem is to find a state feedback gain K such that the closed-loop uncertain system has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) and (17).

We write the closed-loop uncertain system composed of (16) and $u = Kx$ as

$$\begin{cases} x[k+1] \;=\; (A(\alpha) + B_2(\beta)K)\, x[k] + B(\zeta)w[k] \\ \quad\;\; z[k] \;=\; C(\rho)x[k] + D(\psi)w[k] \,. \end{cases} \tag{18}$$

The above system has robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) and (17) if and only if the dual system, represented by $\Big((A(\alpha) + B_2(\beta)K)^T \,,\; C^T(\rho),\; B^T(\zeta),\; D^T(\psi)\Big)$, also has robust $\mathcal{H}_\infty$ performance γ with respect to the same uncertain sets. Hence, according to Theorem 3, the closed-loop uncertain system we obtained has robust $\mathcal{H}_\infty$ performance γ with respect to (2) if there exist positive definite matrices P_{ijkl} and a matrix H such that (G^T is replaced by H for simplicity of notation)

$$\begin{bmatrix} P_{ijkl} & A_{i\beta,K}H & 0 & B_j \\ H^T A_{i\beta,K}^T & H + H^T - P_{ijkl} & H^T C_k^T & 0 \\ 0 & C_k H & \gamma I & D_l \\ B_j^T & 0 & D_l^T & \gamma I \end{bmatrix} > 0 \tag{19}$$

where $A_{i\beta,K} = A_i + B_2(\beta)K$. Setting $KH = L$ in (19) and using the convexity of $B_2(\beta)$ in (17), we obtain the following theorem.

Theorem 4. The system (16) is stabilizable with robust $\mathcal{H}_\infty$ performance γ with respect to the uncertain sets (2) and (17) by state feedback, if there exist positive definite matrices P_{ijklm} and a matrix H such that the LMI

$$\begin{bmatrix} P_{ijklm} & A_i H + B_{2m}L & 0 & B_j \\ (A_i H + B_{2m}L)^T & H + H^T - P_{ijklm} & H^T C_k^T & 0 \\ 0 & C_k H & \gamma I & D_l \\ B_j^T & 0 & D_l^T & \gamma I \end{bmatrix} > 0 \tag{20}$$

holds for all $i = 1, \cdots, N_A$; $j = 1, \cdots, N_B$; $k = 1, \cdots, N_C$; $l = 1, \cdots, N_D$; and $m = 1, \cdots, M$. If this is the case, one robust $\mathcal{H}_\infty$ state feedback gain is given by

$$K = LH^{-1} \,. \tag{21}$$

Remark 2. The nonsingularity of H in Theorem 4 is implied by the $(2,2)$-th diagonal block of (20) since $H + H^T - P_{ijklm} > 0$ and $P_{ijklm} > 0$. □

Finally, we give an example to demonstrate the above result.

Example. Consider the system (18) where only uncertain $A(\alpha)$ exists while all other matrices are constant as

$$\begin{aligned} & A_1 = \begin{bmatrix} 0.3 & -0.3 \\ 0.6 & -0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.8 \\ -0.7 & -0.1 \end{bmatrix} \\ & B(\zeta) = B_2(\beta) \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C(\rho) \equiv \begin{bmatrix} 1 & 0 \end{bmatrix} \\ & D(\psi) \equiv 0 \,. \end{aligned} \tag{22}$$

Our objective is to design a state feedback $u = Kx$ such that the closed-loop uncertain system has robust $\mathcal{H}_\infty$ performance $\gamma = 1.75$. According to Theorem 4, we solve the following two LMIs ($i = 1, 2$)

$$\begin{bmatrix} P_i & A_i H + B_2 L & 0 & B \\ (A_i H + B_2 L)^T & H + H^T - P_i & H^T C^T & 0 \\ 0 & CH & \gamma I & D \\ B^T & 0 & D^T & \gamma I \end{bmatrix} > 0 \tag{23}$$

with respect to $P_1 > 0 \,, P_2 > 0$ and $H \,, L$. Using LMI-Toolbox of MATLAB [11], we obtain

$$\begin{aligned} & P_1 = \begin{bmatrix} 0.99 & 0.29 \\ 0.29 & 1.01 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.94 & -0.19 \\ -0.19 & 1.06 \end{bmatrix} \\ & H = \begin{bmatrix} 2.08 & 0.24 \\ 0.06 & 1.08 \end{bmatrix}, \qquad L = \begin{bmatrix} -0.64 & -0.37 \end{bmatrix} . \end{aligned} \tag{24}$$

Thus, the state feedback gain is computed as

$$K = LH^{-1} = \begin{bmatrix} -0.30 & -0.28 \end{bmatrix}, \qquad (25)$$

and the parameter-dependent Lyapunov matrix is given by $\alpha P_1 + (1-\alpha)P_2$, where $0 \leq \alpha \leq 1$.

However, when we set $P_1 = P_2 = G$ to solve (23) with the same γ, which means solving the control problem by a single Lyapunov matrix, we found that the LMIs are not feasible. This fact tells us that Theorem 3 and Theorem 4 provide less conservative conditions than the existing approach that uses single Lyapunov matrix (or parameter-independent Lyapunov matrix).

4. Conclusion

In this paper, we have derived a new robust $\mathcal{H}_\infty$ performance condition for uncertain discrete-time systems with convex polytopic uncertainties, by using parameter-dependent Lyapunov matrix approach. The condition is expressed as a set of LMIs for the vertix matrices of the polytopes. We have shown that the new condition provides less conservative result than the existing ones which use single Lyapunov matrix. We have also shown that the robust $\mathcal{H}_\infty$ state feedback design problem for such uncertain discrete-time systems can be easily dealt with using the approach. We suggest that the key point of establishing a kind of decoupling between the Lyapunov matrix and the system matrices should be effective for other kind of robust control problems of discrete-time systems with convex polytopic uncertainties.

Acknowledgements. The authors would like to thank Professor Panos J. Antsaklis, the University of Notre Dame, for his valuable comments. This research has been supported in part by the Japan Ministry of Education, Sciences and Culture under Grants-in-Aid for Scientific Research (B) 15760320.

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