# Static Output Feedback Model Predictive Tracking Control for Linear Systems with Uncertainty

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**Abstract:** In this paper, we present static output feedback model predictive tracking control for linear system with uncertainty. The proposed control law is based on integral action form to provide zero offset for constant command signals and the closed loop stability is guaranteed under linear matrix inequality conditions on the terminal weighting matrix using the decreasing monotonicity property of the performance index. Through simulation examples, we illustrate that the proposed schemes can be appropriate tracking controllers for uncertain system.

**Keywords:** Static output feedback, Model predictive tracking control, Uncertainty, LMI

## 1. Introduction

Model predictive control(MPC) has received much attention in control societies because of its good tracking performance and many applications to industrial processing systems[3]. In the model predictive control strategy, at each sample step, an optimal sequence of control inputs, which minimizes an open loop cost function, is computed. The optimization problem also includes hard constraints on the inputs and soft constraints on the outputs or states.The optimization problem is given by minimization of a linear cost function subject to linear matrix inequalities (LMIs). Because the model predictive control technique can easily handle time varying tracking commands, input and output constraints and so on, it has been widely investigated in academia and in industry [7].

In the output-tracking problem, the cost function is defined as the sum of the squared differences between the predicted output at sampling instants and the output reference value. The system outputs should follow the external command signals.

In this paper, we present a static output feedback model predictive tracking control law for linear uncertain systems. We adopt a static output feedback form for the control law. The control law is obtained numerically from a finite horizon output feedback robust optimal tracking control problem. The output feedback model predictive tracking control with integral actions provides zero offset for constant command signal and the closed loop stability is guaranteed under linear matrix inequality conditions on the terminal weighting matrix using the decreasing monotonicity property of the performance. Through simulation examples, we show that the proposed schemes can be appropriate tracking controllers for uncertain time-invariant systems.

## 2. Problem Statement and Preliminaries

Let us consider the following system

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k)+B_p p(k) \\ y(k) &= cx(k) \\ q(k) &= C_q x(k)+D_q u(k) \qquad (1) \\ p(k) &= \Delta_k q(k) \\ \|\Delta_k\| &\leq 1 \end{aligned}$$

where $A, B, C$ and $D$ are the system matrices, $x(k) \in R^n$ is the state, $u(k) \in R^m$ and $y(k) \in R^l$ are the control input and output of the linear block respectively. Here $\Delta_k$ represent some nonlinearities of the plant or parameters, that are unknown, unmodeled or neglected.

For the system (1), we solve the following optimization problem at each time $k$:

$$\min_{P,u} J(k,k+N) \qquad (2)$$

where

$$\begin{aligned} J(k,k+N) &= \sum_{i=k}^{k+N-1}[\tilde{y}^T(i)Q\tilde{y}(i)+u^T(i)Ru(i)] \\ &\quad +\tilde{x}^T(k+N)P_f(k+N)\tilde{x}(k+N) \qquad (3) \end{aligned}$$

where $\tilde{y}(i) = y(k+i|k) - y_r(k+i|k)$, $\tilde{x}(i) = x(k+i|k) - x_r(k+i|k)$, $y_r$ is the desired output reference, $x_r$ is desired state reference, $Q = G^TG \in R^{l\times l}$, $R \in R^{m\times m}$ and $P_f = P_f^T$ are positive definite matrices.

$P_f$ is an important variable for the asymptotic property and closed-loop stability of the receding horizon tracking control (RHTC). Recently for time-varying systems, the terminal inequality condition for implementation of the stabilizing controller is proposed in [8].

So the closed-loop stability of the RHTC is guaranteed under the above terminal inequality condition:

$$\begin{aligned} P_f(i) &\geq C^T(i)Q(i)C(i)+F^T(i)R(i)F(i)+ \\ &\quad (A(i)-B(i)F(i))^TP_f(i)(A(i)-B(i)F(i)) \qquad (4) \end{aligned}$$

In this paper, we consider integral action form because it provides zero-offset for constant reference signals. Consider the cost (3) with $\delta u(k)$ instead of $u(k)$. To derive the control law minimizing the cost (3), we transform the model (1) into the following incremental model:

$$x^e(k+1) = A^e x^e(k)+B^e\delta u(k)+B_p^e\delta p(k)$$

$$
\begin{aligned}
y(k) &= c^e x^e(k) \\
\delta q(k) &= C_q^e x^e(k) + D_q \delta u(k) \\
\delta p(k) &= \Delta_k \delta q(k) \\
\|\Delta_k\| &\leq 1
\end{aligned} \quad (5)
$$

where
$x^e(k) = \begin{bmatrix} y(k) \\ \delta x(k) \end{bmatrix}, A^e = \begin{bmatrix} I & CA \\ 0 & A \end{bmatrix}, B^e = \begin{bmatrix} CB \\ B \end{bmatrix},$
$C^e = \begin{bmatrix} I & 0 \end{bmatrix},\ D_p^e = \begin{bmatrix} CD_p \\ D_p \end{bmatrix},\ C_q^e = \begin{bmatrix} 0 & C_q \end{bmatrix}$
$\delta x(k) = x(k+1) - x(k),\quad \delta u(k) = u(k+1) - u(k)$ ,
$\delta y(k) = y(k+1) - y(k),\quad \delta q(k) = q(k+1) - q(k),$
$\delta p(k) = p(k+1) - p(k)$
The objective of this paper is to find a static type output feedback model predictive tracking controller which stabilizes (1) and makes the output of the plant follow step reference signals.
The controller considered in this paper has the following structure

$$
\delta u(k) = F(k)z(k) + N(k) \quad (6)
$$

where $F(k)$ and $N(k)$ are design variables.

## 3. Static Output Feedback Tracking Control for Uncertainty system

Now we change system and performance index for tracking problem. We augment system states and command signals to derive the static output feedback model predictive tracking control law.

$$
\begin{aligned}
\bar{x}(k+1) &= \bar{A}(k)\bar{x}(k) + \bar{B}_p p(k) \\
q(k) &= E(k)\bar{x}(k)
\end{aligned} \quad (7)
$$

where
$\bar{x}(k) = \begin{bmatrix} x^e(k) \\ 1 \end{bmatrix}, \bar{A}(k) = \begin{bmatrix} A^e + B^e F(k) C^e & B^e N(k) \\ 0 & 1 \end{bmatrix},$
$\bar{B}_p(k) = \begin{bmatrix} B_P^e \\ 1 \end{bmatrix}, E(k) = \begin{bmatrix} C_q^e + D_q F(k) C^e & D_q N(K) \end{bmatrix}$

And the performance index is presented as follows:

$$
\begin{aligned}
J(k, k+N) &= \sum_{i=k}^{k+N-1} [\bar{x}(i)^T \bar{Q}(i) \bar{x}(i)] \\
&+ \bar{x}(k+N)^T \bar{P}(k+N) \bar{x}(k+N)
\end{aligned} \quad (8)
$$

where $\bar{Q}(i) = \begin{bmatrix} \bar{Q}_{11}(i) & \bar{Q}_{12}(i) \\ \bar{Q}_{21}(i) & \bar{Q}_{22}(i) \end{bmatrix},$
$\bar{Q}_{11}(i) := C^T Q(i) C + C^T F^T(i) R F(i) C,$
$\bar{Q}_{21}(i) := N^T(i) R F(i) C - y_r^T(i) Q(i) C,$
$\bar{Q}_{22}(i) := y_r^T(i) Q(i) y_r(i) + N^T(i) R N(i),$
$\bar{P}(k+N) = \begin{bmatrix} \bar{P}_{11}(k+N) & \bar{P}_{12}(k+N) \\ \bar{P}_{21}(k+N) & \bar{P}_{22}(k+N) \end{bmatrix},$
$\bar{P}_{11}(k+N) := P(k+N),$
$\bar{P}_{12}(k+N) := -P(k+N) C^T y_r(k+N),$
$\bar{P}_{22}(k+N) := y_r^T(k+N) C^{-T} P(k+N) C^{-1} y_r(k+N).$
Now, we derive the control law from the augmented system (7) with the performance index (8).
And using the following theorem, we solve the minimization problem.

**Theorem 1:** Assume that there exists $\bar{P}(i)$ satisfying such that

$$
\begin{bmatrix} \bar{x}(i) \\ p(i) \end{bmatrix}^T \begin{bmatrix} X_{11}^* & X_{12}^* \\ X_{21}^* & X_{22}^* \end{bmatrix} \begin{bmatrix} \bar{x}(i) \\ p(i) \end{bmatrix} < 0 \quad (9)
$$
$$
\begin{aligned}
&X_{11}^* := \bar{A}^T(i)\bar{P}(i+1)\bar{A}(i) + \bar{Q}(i) - \bar{P}(i), \\
&X_{21}^* := \bar{B}_p{}^T \bar{P}(i+1)\bar{A}, X_{22}^* := \bar{B}_p{}^T \bar{P}(i+1)\bar{B}_p \\
&\quad i = k, ..., k+N-1.
\end{aligned}
$$

for all $\bar{x}(i)$ satisfying

$$
\begin{bmatrix} \bar{x}(i) \\ p(i) \end{bmatrix}^T \begin{bmatrix} E^T(i)E(i) & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \bar{x}(i) \\ p(i) \end{bmatrix} \geq 0 \quad (10)
$$

then

$$
J(k) < \bar{x}^T(k)\bar{P}(k)\bar{x}(k) \quad (11)
$$

**Proof:** Along the state trajectory of (7), we can obtain the following relation

$$
\begin{aligned}
&J(k, k+N) - \bar{x}(k)\bar{P}(k)\bar{x}(k) \\
&= \sum_{i=k}^{k+N-1} [\bar{x}(i)^T \bar{Q}(i)\bar{x}(i) + \bar{x}(i+1)^T \bar{P}(i+1)\bar{x}(i+1) \\
&\quad - \bar{x}(i)^T \bar{P}(i)\bar{x}(i)]
\end{aligned} \quad (12)
$$

Consequently, if we select $\bar{P}(i)$ for each $i = k+N-1, ..., 0$ such that

$$
\begin{aligned}
&\bar{x}(i)^T \bar{Q}(i)\bar{x}(i) + \bar{x}(i+1)^T \bar{P}(i+1)\bar{x}(i+1) \\
&\quad - \bar{x}(i)^T \bar{P}(i)\bar{x}(i) < 0
\end{aligned} \quad (13)
$$

for all uncertainties $\Delta(i)$ (The inequality (12) is equivalent to (9), (10)), then the finite horizon performance index is bounded by

$$
J(k) < \bar{x}(k)^T \bar{P}(k)\bar{x}(k) \quad (14)
$$

■

Thus, the upper bound of the performance index $J(k)$ depends on $\bar{P}$. To minimize the performance index, minimize $trace(\bar{P}(i))$ satisfying the inequality(9),(10) with given $\bar{P}(i+1)$ for the fixed finite horizon.

We derive the static output feedback model predictive tracking control law based on the algorithm. First, We assume $y_r = 0$ and then we find $P(k+N) = P_f = \bar{P}_{11}(k+N)$.
In this paper $\bar{P}_{11}(k+N)$ means(1,1) block of $\bar{P}(k+N)$.
In order to find $P(k+N) = P_f$. Using the `S-procedure` for (9) and (10) following inequivalent condition should be satisfied :

$$
\begin{bmatrix} X_{11}^* + \lambda E^T(i)E(i) & \bar{A}^T(i)\bar{P}(i+1)\bar{B}_p \\ \bar{B}_p^T \bar{P}(i+1)\bar{A}(i) & \bar{B}_p^T \bar{P}(i+1)\bar{B}_p - \lambda I \end{bmatrix} < 0,\ \lambda > 0 \quad (15)
$$

The $P_f$ is obtained from the solution of the following linear performance minimization problem from the condition (15):

$$\min_{(Y,S,\lambda)} \quad P_f$$

subject to $P_f > 0$ , $\lambda > 0$ and

$$\begin{bmatrix} -Y & * & * & * & * \\ S & -R^{-1} & * & * & * \\ AY+BS & 0 & M_{33} & * & * \\ C_qY+D_qS & 0 & 0 & -\lambda^{-1}I & * \\ CY & 0 & 0 & 0 & -Q^{-1} \end{bmatrix} < 0, \quad (16)$$

where $Y = P_f{}^{-1}$ and $S = FCY$, $M_{33} := -Y + \lambda^{-1}B_pB_p^T$

Actually, the calculated $P(k+N) = P_f$ is terminal weighing matrix. we find $\bar{P}(k+i)$ with given $\bar{P}(k+i+1)$ from $i = N-1$ to 0. First, we need to reconstruct $\bar{P}(k+i)$ from the following relation:

$$\bar{P}(k+N) = \begin{bmatrix} \bar{P}_{11}(k+N) & \bar{P}_{12}(k+N) \\ \bar{P}_{21}(k+N) & \bar{P}_{22}(k+N) \end{bmatrix},$$

where $\bar{P}_{11}(k+N) := P(k+N)$, $\bar{P}_{12}(k+N) := -P(k+N)C^Ty_r(k+N)$ and $\bar{P}_{22}(k+N) := y_r^T(k+N)C^{-T}P(k+N)C^{-1}y_r(k+N)$.

Next, we find $\bar{P}(i)$ with given a matrix $\bar{P}(i+1)$ from $i = k+N-1$ to $k$. By the `Schur's complement`, the condition (15) can be changed to the following two inequalities:

$$\bar{B}_p^T\bar{P}(i+1)\bar{B}_p - \lambda I < 0 \quad (17)$$

$$\begin{bmatrix} \bar{Q}(i)-\bar{P}(i) & E^T(i) & \bar{A}^T(i) \\ E(i) & -\lambda^{-1}I & 0 \\ \bar{A}(i) & 0 & X_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \Phi_{11} & * & * & * & * & * \\ \Phi_{21} & \Phi_{22} & * & * & * & * \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & * & * & * \\ \Phi_{41} & \Phi_{42} & 0 & \Phi_{44} & * & * \\ \Phi_{51} & \Phi_{52} & 0 & 0 & \Phi_{55} & * \\ \Phi_{61} & \Phi_{62} & 0 & 0 & \Phi_{65} & \Phi_{66} \end{bmatrix} < 0, \quad (18)$$

Where $\Phi_{ij}$

$\Phi_{11} := C^{eT}Q(i)C^e - \bar{P}_{11}(i)$,
$\Phi_{21} := -y_r^T(i)Q(i)C^e - \bar{P}_{21}(i)$,
$\Phi_{22} := y_r^T(i)Q(i)y_r(i) - \bar{P}_{22}(i)$,
$\Phi_{31} := F(i)C^e$,
$\Phi_{32} := N(i)$,
$\Phi_{33} := -R^{-1}$,
$\Phi_{41} := C_q^e + D_qF(i)C^e$,
$\Phi_{42} := D_qN(i)$,
$\Phi_{44} := -\lambda^{-1}$,
$\Phi_{51} := \bar{P}_{11}^T(i+1)A^e + \bar{P}_{11}^T(i+1)B^eF(i)C^e$,
$\Phi_{52} := \bar{P}_{11}^T(i+1)B^eN(i) + \bar{P}_{21}(i+1)$,
$\Phi_{55} := -\bar{P}_{11}(i+1) + \lambda^{-1}\bar{P}_{11}^T(i+1)B_p^eB_p^{eT}\bar{P}_{11}(i+1)$,
$\Phi_{61} := \bar{P}_{21}(i+1)A^e + \bar{P}_{21}(i+1)B^eF(i)C^e$,
$\Phi_{62} := \bar{P}_{21}(i+1)B^eN(i) + \bar{P}_{22}^T(i+1)$,
$\Phi_{65} := -\bar{P}_{21}(i+1) + \lambda^{-1}\bar{P}_{21}(i+1)B_p^eB_p^{eT}\bar{P}_{11}(i+1)$,
$\Phi_{66} := -\bar{P}_{22}(i+1) + \lambda^{-1}\bar{P}_{21}(i+1)B_p^eB_p^{eT}\bar{P}_{12}(i+1)$,

$$\bar{P}(i+1) := \begin{bmatrix} \bar{P}_{11}(i+1) & \bar{P}_{21}^T(i+1) \\ \bar{P}_{21}(i+1) & \bar{P}_{22}(i+1) \end{bmatrix},$$

**Theorem 2:** Assume that the the problem (4) subject to (9) for $i = k+N$ is feasible at each time $k$, if there exists$P_f(i) = P_f^T(i) > 0$, $F(i)$ and $N(i)$ satisfying the inequality (18), then the tracking-error of the closed-loop system (7) goes to zero.

Furthermore, the minimum value of the performance index can be obtained by the following optimization programming:

$$\min_{\bar{P}(i),F(i),N(i),\lambda} \quad trace(P_f(i)) \quad (19)$$

subject to (17),(18).

**Proof:** $\bar{P}^e(i+1)$ given from the inequality (9). By using the Schur compliment, the inequality conditon (17), (18) can be easily obtained. ■

Therefore, we can obtain the static output feedback model predictive tracking controller variable $F(k)$ and $N(k)$ to minimize the performance index recursively.

## 4. Simulation

$$A = \begin{bmatrix} 0.15 & 0.05 & 0.02 \\ -0.1 & 0.2 & 0.05 \\ 0.03 & -0.11 & 0.5-0.1a(k) \end{bmatrix}, \; B = \begin{bmatrix} -0.2 \\ -0.06 \\ 0.02 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0 & -1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \end{bmatrix},$$

$0.1 \le a(k) \le 5$, Alternatively, if we define

$$\delta(k) = \frac{a(k)-0.27}{0.21}, \; |\delta| \le 1$$

$$A = \begin{bmatrix} 0.15 & 0.05 & 0.02 \\ -0.1 & 0.2 & 0.05 \\ 0.03 & -0.11 & 0.21 \end{bmatrix}, \; B_p = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.01 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0.1 & 0.05 & -0.2 \end{bmatrix}, \; D_q = \begin{bmatrix} 0.03 \end{bmatrix},$$

We choose simulation parameters as follows: $Q = 10I$, $R = 0.01I$, $N = 8$.

The performance of the proposed controller is shown in figure 1-2, where we can know that the proposed controller yields a nice tracking performance.

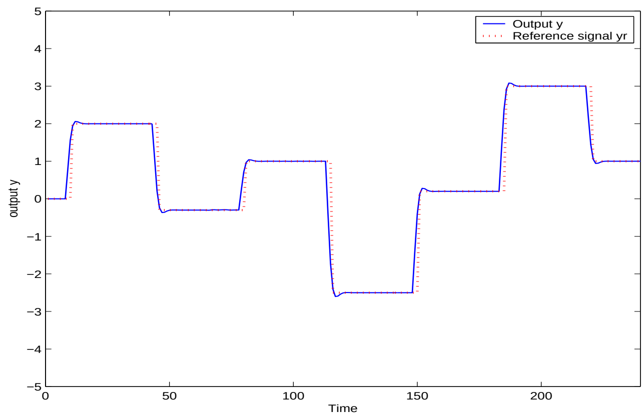


Fig. 1. Simulation results : output $y$

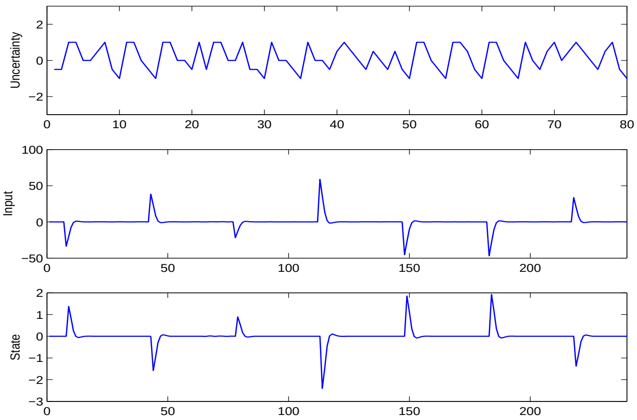

Fig. 2.   Uncertainty, Input, State

## 5. Conclusion

In this paper, we proposed static output feedback model predictive tracking control for linear system with uncertainty. We adopted a static output feedback form for the control law rather than an observer based form. Because hardly solved analytically that the closed loop stability was guaranteed, We solved that under linear matrix inequality(LMI) conditions on the terminal weighting matrix using the decreasing monotonicity property of the performance index in uncertain time-invariant system. A numerical example demonstrated the nice performance of the proposed method for uncertain time-invariant system.

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