A Novel Eigenstructure Assignment for Linear Systems with Probabilistic Uncertainties

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Abstract: In this paper, S(stochastic)-eigenvalue concept and its S-eigenvector for linear continuous-time systems with probabilistic uncertainties are proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the dynamic model of a plant. S-eigenstructure assignment scheme via the Sylvester equation approach based on the S-eigenvalue concept is also proposed. The proposed design scheme is applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure.

Keywords: Probabilistic Parameter Variation, Gaussian Distribution, S-Eigenvalue, S-Eigenvector, S-Stability, S-Eigenstructure Assignment

1. Introduction

In recent years, eigenstructure assignment has been applied to the design of various kinds of practical multivariable control systems, e.g., helicopters, aircraft, missiles, generator, voltage regulators and mechanical systems[1]. The specified effect of eigenstructure assignment is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. In general, the speed of response is determined by the assigned eigenvalues whereas the shape of the response is furnished by the assigned eigenvectors[2]. The eigenstructure assignment technique is used to design flight control laws for aircraft with many control efforts, and the technique together with suitable feedforward design can achieve static decoupling with internal stability, which is an important requirement in many flight control system[3]-[5].

In eigenstructure assignment techniques, the design parameters are the desired closed-loop eigenvalues and specified elements of the closed-loop eigenvectors. Once the design parameters are specified, the feedback control gains are determined. Therefore, given a set of specifications, the feedback control gains will provide the desired closed-loop transient response (or come as close to it as possible within the system constraints), but they might result in a system with poor stability robustness[6], *i.e.*, a small change in the plant dynamics may cause the closed-loop system to go unstable. The designer is then faced with the dilemma of how to change the design specifications such that the resulting feedback system will also provide adequate stability robustness. Note that, in general, the designer does have a certain amount of freedom in choosing the design specifications. The designer rarely wants an exact value for a closed-loop eigenvalue or exact shape for a corresponding eigenvector. The specifications are rather in terms of desired regions for the closed-loop eigenvalues and acceptable sets of eigenvector shapes. The general eigenstructure assignment methodologies cannot guarantee stability robustness to parameter variations of a system. This problem is still unsolved, thus it is worthwhile to explore the extension of the conventional eigenstructure assignment technique.

In this paper, first, a novel eigenvalue and its corresponding eigenvector concept for linear systems with probabilistic uncertainties are proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the plant's dynamic model. The probability that all eigenvalues lie in the open left-half s plane is the scalar measure of robustness. Second, the stability based on the proposed concept is presented on the appropriate random characteristics of perturbations to maintain the proper stability behavior of the overall system. Third, S-eigenstructure assignment scheme via a Sylvester equation approach based on the S-eigenvalue concept is proposed. The proposed design scheme is applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure.

2. S-Eigenvalue/Eigenvector

By $M_n(\mathbf{F})$ we denote the n-by-n matrices over a field $\mathbf{F}$, usually the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. Also the set (vector space) of all real-entried (respectively complex-entried) n vectors is denoted by $\mathbf{R}^n$ (respectively $\mathbf{C}^n$), both interpreted as column vectors[7]. All matrices are with compatible dimensions if they are not explicitly stated.

Consider the linear time-invariant system with probabilistic parameters as follows:

$$\delta x = F(p)x + G(p)u, \tag{1}$$

$$u = u_c - K(p)x \tag{2}$$

where δx represents $\dot{x}(t)$ for continuous systems and $x(t+1)$ for discrete systems, $x \in \mathbf{R}^n$ the state vector, $u \in \mathbf{R}^m$ the control input vector. The matrices $F(p) \in \mathbf{R}^{n\times n}$ and $G(p) \in \mathbf{R}^{n\times m}$ are system and input matrices that may be Gaussian random parameter, p. u_c is a command input vector, and, for simplicity, the $(m \times n)$ control gain matrix, $K(p)$, is assumed to be known. A linear state-feedback control law (2) is applied to the continuous system (1), then the closed-loop system representation is given by

$$\begin{aligned} \dot{x}(t) &= (F(p) - G(p)K(p))\, x(t) \\ &\triangleq A(p)x(t) \end{aligned} \tag{3}$$

where, the n eigenvalues, $\rho_i(p)$, of the matrix $[F(p) - G(p)K(p)]$ determine closed-loop stability and can be determined as the roots of the determinant equation

$$\det\left(sI_n - [F(p) - G(p)K(p)]\right) = 0 \tag{4}$$

The n eigenvalues of the closed-loop system can be represented as the mean of eigenvalues plus the perturbation terms, respectively.

$$\rho_i(p) = \rho_i^E + \tilde{\rho}_i(p), \; i = 1, \cdots, n \tag{5}$$

where, $\rho_i^E = \mathrm{E}[\rho_i(p)]$ and $\tilde{\rho}_i(p)$ denote the mean of ith eigenvalue and the variation from the mean of ith eigenvalue, respectively. The perturbation term in *Eq.*(5) reflects the eigenvalues variation due to the probabilistic parameter variation of the system matrix. Because root loci for individual parameter variations would follow classical configurations of root locus construction, with the heaviest density of roots in the vicinities of the nominal roots, the probabilistic distribution of eigenvalues may be assumed to be the Gaussian distribution. The basic justification of this statement is embodied in the central limit theorem: one of its numerous precise statements(differing in specific assumptions and details, but all essentially the same) is now stated.

Theorem 1. Central Limit Theorem[8]
Let $\left\{\rho_i^k(p) \,|\, i = 1, \cdots, n\right\}$ be a set of eigenvalues that are calculated from the production set, $\left\{A^k(p)\right\}_{k=1}^N$, of a linear time-invariant system with probabilistic parameter variations on a large scale. Also, let each element of $\left\{\tilde{\rho}_i^k(p) = \rho_i^k(p) - \rho_i^E\right\}_{k=1}^N$ be a n-vector which are identically distributed with means and covariance matrices m^k and P^k, respectively. Define the random vector y_N as their sum:

$$y_N = \sum_{i=1}^{N} \tilde{\rho}_i^k(p)$$

and also define $\tilde{\rho}_i(p)$ as the (zero-mean) normalized sum random variable:

$$\tilde{\rho}_i(p) = \left[P_{y_N y_N}\right]^{-1/2} \left[y_N - \mathrm{E}\left[y_N\right]\right]$$

where

$$\mathrm{E}\left[y_N\right] = \sum_{k=1}^{N} m^k, \quad P_{y_N y_N} = \sum_{k=1}^{N} P^K, \text{ and } P^{-1/2} = \left(P^{1/2}\right)^{-1}$$

where $P^{1/2}$ is defined as the n-by-n matrix such that $P^{1/2}(P^{1/2})^T$. Then, in the limit as $N \to \infty$, $\tilde{\rho}_i(p)$ becomes a zero-mean Gaussian random n-vector with a covariance matrix equal to identity matrix:

$$\lim_{N\to\infty} f_{\tilde{\rho}_i(p)}(\zeta) = \left[(2\pi)^{n/2}\right] \exp\left\{-\frac{1}{2}\zeta^T\zeta\right\} \qquad \square$$

The proof is going to show that $\tilde{\rho}_i(p)$ converges in distribution to a random variable having a standard normal distribution by showing that the moment generating function of $\tilde{\rho}_i(p)$ converges to the moment generating function of the standard normal distribution. The theorem states that if the eigenvalues are generated as the sum of eigenvalues of many identical system, the probabilistic distribution of eigenvalues (5) approaches a Gaussian distribution as more eigenvalues are summed. The eigenvalue-eigenvector equation of the closed-loop system (3) with Gaussian distribution eigenvalues can be defined as follows:

Definition 1. Let $A(p) \in M_n$ and $\phi(p) \in \mathbf{C}^n$. Consider

$$A(p)\phi(p) = \rho(p)\phi(p), \quad \phi(p) \neq \mathbf{0}, \tag{6}$$

where $\rho(p)$ is a scalar. If a scalar $\rho(p)$ and a nonzero vector $\phi(p)$ happen to satisfy this equation, then $\rho(p)$ is called an "*S-eigenvalue*" of $A(p)$ and $\phi(p)$ is called an "*S-eigenvector*" of $A(p)$ associated with $\rho(p)$. Notice the two occur inextricably as a pair, and that an S-eigenvector cannot be the zero vector.

Suppose that n-eigenvalues can be plotted l times for time interval $[t_0, \; t_f]$ on the complex plane, then $n \times l$-eigenvalues may be plotted on the complex plane. If all of $n \times l$-eigenvalues lie in the left-half s plane, then the stability of closed-loop system is guaranteed. But, if some eigenvalues lie in the right-half s plane, then the closed-loop system has the probability of instability. Thus, the probability that all of these eigenvalues lie in the left-half s plane is the scalar measure of robustness-stability. The density of these eigenvalues depicts the likelihood that eigenvalues vary from their mean values, this can be gained by plotting the probability density function corresponding eigenvalue on a three-dimensional complex plane. From Theorem 1, the probability density function of S-eigenvalue is assumed to be the Gaussian distribution as follows:

Definition 2. Let $A(p) \in M_n$ be a closed-loop system. For $A(p)$, the probability density function corresponding to the S-eigenvalue in a complex plane $\mathbf{C}$ is defined by:

- Case 1: Complex conjugate eigenvalue

$$\mathrm{pr}\left(\rho(p)\right) = \frac{1}{2\pi\left|\Sigma\right|^{1/2}} \exp\left[-\frac{1}{2}\tilde{P}^T(p)\Sigma^{-1}\tilde{P}(p)\right] \tag{7}$$

- Case 2: Real eigenvalue

$$\mathrm{pr}\left(\sigma(p)\right) = \frac{1}{\sqrt{2\pi}\sigma_{\tilde{\sigma}}} \exp\left[-\frac{1}{2\sigma_{\tilde{\sigma}}^2}\left(\sigma(p) - \mathrm{E}\left[\sigma(p)\right]\right)^2\right] \tag{8}$$

with $\tilde{P}^T(p) = \left[\begin{array}{cc} \tilde{\sigma}(p) & \tilde{\omega}(p) \end{array}\right]^T$ and $\Sigma = \mathrm{diag}\left(\sigma_{\tilde{\sigma}}^2, \; \sigma_{\tilde{\omega}}^2\right)$, where Σ is a positive $(n \times n)$ matrix, $|\cdot|$ denotes the determinant of a matrix, and $\exp[\cdot]$ denotes exponential. The quantities

$$P^E = \mathrm{E}[P(p)] = \mathrm{E}\left[\begin{array}{c} \sigma(p) \\ \omega(p) \end{array}\right]$$

and

$$\Sigma = \mathrm{E}\left[(P(p) - P^E)(P(p) - P^E)^T\right]$$

are the mean and covariance of the vector $P(p)$, respectively.

Next, define the spectrum corresponding to S-eigenvalue as following:

Definition 3. The set of all $\rho(p) \in \mathbf{C}$ that are S-eigenvalues of $A(p) \in M_n$ is called the "*S-spectrum (stochastic-spectrum)*" of $A(p)$ and is denoted by $\sigma(A(p))$. The spectral radius of $A(p)$ is the non-negative real number $\rho(A) = \max\{|\rho(t)| : \rho(p) \in \sigma(A(p))\}$. This is the radius of the smallest disc centered at the origin in the complex plane that includes all the S-eigenvalues of $A(p)$.

And, the S-eigenvector corresponding to S-eigenvalue can be stated as following theorem:

Theorem 2. Let $A(p) \in M_n$. For a given $\rho(p) \in \sigma(A(p))$, the set of all S-eigenvectors $\phi_i(p) \in \mathbf{C}^n$ satisfying $A_i(p)\phi_i(p) = \rho_i(p)\phi_i(p)$ is called the "*S-eigenspace (stochastic-eigenspace)*" of $A(p)$ corresponding to the S-eigenvalues.

$$\phi_i(p) = \gamma_i(p)\phi_i^E \tag{9}$$

where $\gamma_i(p) = \left\{I + (\rho_i(p)I - A(p))^{-1}(\tilde{A}(p) - \tilde{\rho}_i(p)I)\right\}$, $\phi_i^E = \phi_i(p) - \tilde{\phi}_i(p)$, $\phi_i^E = \mathrm{E}[\phi_i(p)]$, and $\mathrm{VAR}\left[\tilde{\phi}_i(p)\right] = \phi_i^E\left(\phi_i^E\right)^T$.

Proof: Substitute $\rho_i(p) = \rho_i^E + \tilde{\rho}_i(p)$, $\phi_i(p) = \phi_i^E + \tilde{\phi}_i(p)$, and $A(p) = A^E + \tilde{A}(p)$ for *Eq.* (6),

$$(\rho_i^E I + \tilde{\rho}_i(p)I - A^E - \tilde{A}(p))(\phi_i^E + \tilde{\phi}_i(p)) = 0$$

and,

$$\begin{aligned}&(\rho_i^E I - A^E)\phi_i^E + (\tilde{\rho}_i(p)I - \tilde{A}(p))\phi_i^E \\ &+(\rho_i^E I + \tilde{\rho}_i(p)I - A^E - \tilde{A}(p))\tilde{\phi}_i(p) = 0.\end{aligned}$$

From the characteristics of deterministic eigenvalue problem, $(\rho_i^E I - A^E)\phi_i^E = 0$. The above equation is classified in terms of $\tilde{\phi}_i(p)$ as follows:

$$\tilde{\phi}_i(p) = (\rho_i(p)I - A(p))^{-1}(\tilde{A}(p) - \tilde{\rho}_i(p)I)\phi_i^E$$

Because of $\phi_i(p) = \phi_i^E + \tilde{\phi}_i(p)$, *Eq.* (9) can be obtained. Also, In such a case, the mean and covariance of S-eigenvector can be easily shown as follows:

$$\mathrm{E}[\phi_i(p)] = \phi_i^E$$

$$\begin{aligned}\mathrm{VAR}\left[\tilde{\phi}_i(p)\right] &= \mathrm{E}\left[\phi_i(p)\phi_i^T(p)\right] \\ &= \mathrm{E}\left[\gamma_i(p)\phi_i^E\left(\gamma_i(p)\phi_i^E\right)^T\right] \\ &= \phi_i^E\left(\phi_i^E\right)^T\end{aligned}$$

□

The S-eigenvector to the mean eigenvector may be represented as the following absolute misalignment angle.

$$\theta_i(p) = \cos^{-1}\left(\frac{\left|\phi_i^T(p)\phi_i^E\right|}{\|\phi_i(p)\|_2\,\|\phi_i^E\|_2}\right) \tag{10}$$

where, $\theta_i(p)$ is a linear operator[9] and geometrically identical with $\gamma_i(p)$ of *Eq.* (9). Theorem 2states that S-eigenvector rotating about a fixed, arbitrary nominal eigenvector.

Definition 4. A matrix $\Phi(p) \in \mathbf{C}^{n\times n}$ is called a "*S-modal (stochastic-modal) matrix*" of $A(p) \in M_n$ corresponding to $\rho(p) \in \sigma(A(p))$ if:

$$\Phi(p) = \Gamma(p)\Phi^E$$

where

$$\Gamma(p) = \left[\begin{array}{cccccc}\gamma_1(p) & \gamma_2(p) & \cdots & \gamma_i(p) & \cdots & \gamma_n(p)\end{array}\right],$$

$$\gamma_i(p) = \left\{I + (\rho_i(p)I - A(p))^{-1}(\tilde{A}(p) - \tilde{\rho}_i(p)I)\right\},$$

and

$$\Phi^E = \Phi(p) - \tilde{\Phi}(p).$$

3. S-Stability

It is well known that an LTI system is asymptotically stable if, and only if, deterministic eigenvalues are in the LHP of $\mathbf{C}$. If all of eigenvalues lie in the left-half s plane, then the LHP stability of closed-loop system is guaranteed. But, the S-eigenvalue with the probabilistic distribution does not guarantee the deterministic LHP stability criterion directly, because S-eigenvalues vary with the probabilistic uncertainty. First, in order to deal with probabilistic stability, define S-mean (stochastic-mean) of the S-eigenvalues as follows:

Definition 5. Let $\sigma(p)$ be a real variable of S-eigenvalues on the complex plane $\mathbf{C}$. The "*S-mean (stochastic-mean)*", $\mathrm{sm}(\sigma(p))$, over $\mathbf{C}$ is defined by:

$$\mathrm{sm}(\sigma(p)) = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\sigma(p)dp \tag{11}$$

where $\sigma(p) = \sigma^E + \tilde{\sigma}(p)$, $\sigma^E = \mathrm{E}[\sigma(p)]$, and stochastic integrals of $\tilde{\sigma}(p)$ existed.

Definition 5 is defined to use that the real value of the eigenvalue determines the stability of the plant. The density of these eigenvalues depicts the likelihood that eigenvalues vary from their mean values, these means have only to exist on the left-half plane at least. Next, the probability of stability of the S-eigenvalue is defined as follows:

Definition 6. Let $\mathrm{pr}(\rho(p))$ be a given PDF corresponding to an S-eigenvalue in a complex plane $\mathbf{C}$. If $\mathrm{pr}(\rho(p))$ be a stochastic integrable function on the LHP $(-\infty,\ 0]$, then the "*probability of stability*" of LTI stochastic systems is defined by:

$$\mathrm{S} = \int_{-\infty}^{0}\mathrm{pr}(\rho(p))\,dp \tag{12}$$

where $0 \le \mathrm{S} \le 1$. Notice that the probability of stability in the ergodic sense is given by:

$$\mathrm{S} = \lim_{J\to\infty}\frac{1}{Jn}\sum_{J=1}^{\infty}N(\sigma(p) \le 0) \tag{13}$$

where $N(\cdot)$ is the number of cases for which all elements of $(\cdot)$ are less than or equal to zero, n is the dimension of the system, and J is the number of Monte Carlo evaluation.

Using definitions 5 and 6, the S-stability criterion based on the S-eigenvalue can be stated by the following theorem.

Theorem 3. Let $\rho(p)$ be an S-eigenvalue of $A(p)$. Then the solution to $\det(\rho_i(p)I - A(p)) = 0$ is stochastically stable for all t if and only if:
i) there exists $0 < \sigma_i^E \leq \infty$ such that

$$\mathrm{sm}(\sigma_i(p)) = -\sigma_i^E < 0$$

and moreover,
ii) there exists $\varepsilon > 0$ such that

$$\mathrm{P}\{|1 - \mathrm{S}| > \varepsilon\} \to 0$$

as $J \to \infty$ for $\forall t, t \geq T_0$.

Proof: Condition i) states that the mean of the real value of eigenvalue is less than arbitrary negative value $-\sigma_i^E$. Thus, the core axis of the Gaussian distribution is located on the left-half plane at least. But, though condition i) is guaranteed by itself, the probability of instability remains still. In order to guarantee the stochastic stable, the probability of the instability approaches a zero in the process of repeating the simulations as depict in condition ii). □

In Theorem 3, the sequence of random variables $\{X_n(\xi)\}$ converges in probability(Leon-Garcia, 1994) to the random variables $\{X(\xi)\}$ if, for any $\varepsilon > 0$:

$$\mathrm{P}[X_n(\xi) - X(\xi) > \varepsilon] \to 0 \text{ as } n \to \infty.$$

4. Stochastic-Eigenstructure Assignment

The specified effect of eigenstructure assignment is achieved by assigning a certain set of eigenvalues and an associated set of eigenvectors to the closed-loop system. First, from the previous definitions, the required S-eigenvalues could be established as follows:

$$\rho_i^d(p) = \left(\rho_i^E\right)^d + \tilde{\rho}_i^d(p)$$

where $\mathrm{E}\left[\rho_i^d(p)\right] = \left(\rho_i^E\right)^d$. And, if the mean of the required modal matrix, $\left(\Phi^E\right)^d$, is determined, the required S-modal matrix corresponding to the required S-eivenvalues could be established as follows:

$$\Phi^d(p) = \Gamma^d(p)\left(\Phi^E\right)^d$$

where

$$\Gamma^d(p) = \left[\ \gamma_1^d(p) \quad \gamma_2^d(p) \quad \cdots \quad \gamma_i^d(p) \quad \cdots \quad \gamma_n^d(p)\ \right],$$

$$\gamma_i^d(p) = \left\{I + (\rho_i^d(p)I - F(p))^{-1}(\tilde{F}(p) - \tilde{\rho}_i^d(p)I)\right\}.$$

Our objective is to find the feedback-gain matrix $K(p)$ such that the closed-loop S-eigenvalues are obtained exactly, and that the required S-eigenvectors are assigned to the best possible set of eigenvectors with consistency of statistical fitting procedures.

Theorem 4. For a given set of $F(p)$, $G(p)$ matrices, which are LTI stochastic systems with inputs, and for a S-eigenvalues matrix $\Lambda(p) = \Lambda^E + \tilde{\Lambda}(p)$ and S-modal matrix $\Phi(p) = \Gamma(p)\Phi^E$, a parameter matrix $H(p)$ could be chosen by the following equation:

$$\Lambda^E\Gamma(p)\Phi^E + \tilde{\Lambda}(p)\Gamma(p)\Phi^E - \Gamma(p)\Phi^E F(p) = G(p)H(p) \quad (14)$$

where $\Phi^E = [\,\phi_1^E \quad \phi_2^E \quad \cdots \quad \phi_n^E\,]$, $\Lambda^E = \mathrm{diag}\left(\rho_1^E,\ \rho_2^E,\ \cdots,\ \rho_n^E\right)$, $\tilde{\Lambda}(p) = \mathrm{diag}\,(\,\tilde{\rho}_1(p) \quad \tilde{\rho}_2(p) \quad \cdots \quad \tilde{\rho}_n(p)\,)$ and $H(p) = [\,h_1(p) \quad h_2(p) \quad \cdots \quad h_n(p)\,]$.

Proof. If a state feedback $u(t) = -K(p)x(t)$ is applied to $\dot{x}(t) = F(p)x(t) + G(p)u(t)$, the closed-loop system becomes $\dot{x}(t) = (F(p) - G(p)K(p))x(t)$. The corresponding right S-eigenvalue problem is then defined by:

$$(F(p) - G(p)K(p))\,\phi_i(p) = \rho_i(p)\phi_i(p) \quad (15)$$

where $\phi_i(p)$ is the right S-eigenvector corresponding to the S-eigenvalue $\rho_i(p)$. The parameter vector $h_i(p) \in \mathbf{C}^m$ is defined by:

$$h_i(p) = K(p)\phi_i(p). \quad (16)$$

Then, *Eq.* (15) is put in the form of the Sylvester equation:

$$(F(p) - \rho_i(p)I)\,\phi_i(p) = G(p)h_i(p) \quad (17)$$

or:

$$\left(F(p) - (\rho_i^E + \tilde{\rho}_i(p)I)\right)\gamma_i(p)\phi_i^E = G(p)h_i(p) \quad (18)$$

The matrix form of *Eq.* (18) can be shown as *Eq.* (14). □

Using Theorem 4, we can solve for $K(p)$ from the linear equation:

$$K(p)\Gamma(p)\Phi^E = H(p). \quad (19)$$

where the inverse matrix of $\Gamma(p)\Phi^E\ (= \Phi(p))$ is always existed, $K(p)$ consists of probabilistic elements due to variations of $\Gamma(p)\Phi^E$.

5. Simulation and Results

An example of the application is based on the longitudinal dynamics of an open-loop unstable aircraft[10]. The Forward-Swept-Wing Demonstrator's aerodynamic center is forward of its center of gravity, resulting in static instability. Possible uncertainties in aerodynamic and thrust effect as well as separate dynamic pressure (ρ and V) effects lead to a 12-element parameter vector, $p = [\,\rho \quad V \quad f_{11} \quad f_{12} \quad f_{13} \quad f_{22} \quad f_{32} \quad f_{33} \quad g_{11} \quad g_{12} \quad g_{31} \quad g_{32}\,]$. Velocity($V$) and air-density($\rho$) are modeled as uniform parameter, the remaining terms are kinematics, due to gravity, identically zero or otherwise negligible. Each parameter perturbations are distributed around the nominal value and correlation is assumed to independent on each other. In terms of the element p, $F(p)$ and $G(p)$ are

$$F(p) = \begin{bmatrix} -2gf_{11} & \frac{\rho V^2 f_{12}}{2} & \rho V f_{13} & -g \\ \frac{-45}{V^2} & \frac{\rho V f_{22}}{2} & 1 & 0 \\ 0 & \frac{\rho V^2 f_{32}}{2} & \rho f_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G(p) = \frac{\rho V^2}{2} \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & g_{32} \\ 0 & 0 \end{bmatrix}.$$

The state components represent forward velocity, angle of attack, pitch rate, and pitch angle. Principal control surfaces are the canard control surface and the thrust setting. The mean model and its eigenvalues for the given system are as follows:

$$F^E = \begin{bmatrix} -0.02 & -0.3 & -0.4 & -32.2 \\ -0.001 & -1.2 & 1 & 0 \\ 0 & 18. & -0.6 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$G^E = \begin{bmatrix} -0.04 & 35. \\ 0 & 0 \\ 0.2 & -0.2 \\ 0 & 0 \end{bmatrix},$$

$$\rho^E_{1-4} = [-5.1535 \quad -0.0102 \pm 0.057i \quad 3.3539].$$

For illustration, ρ and V are 10% standard deviation Gaussian uncertainties, and the remaining elements of p are subjected to independent 30% standard deviation Gaussian uncertainties. The open-loop eigenvalues distribution of the flight control application on two-dimensional complex plane is shown in Fig. 1.

Let the desired eigenvalues of the closed-loop system so that the natural frequency of the remaining eigenvalue can be three or five times as large as the one of a dominant eigenvalue as follows:

$$\left(\rho_1^E\right)^d = -5.1535, \;\; \sigma^d_{\tilde{\sigma}_1} = 0.6492$$

$$\left(\rho_2^E\right)^d = -4, \;\; \sigma^d_{\tilde{\sigma}_2} = 0.3525$$

$$\left(\rho_3^E\right)^d = -0.5 + i, \;\; \Sigma_3^d = diag(0.0102, \; 0.0099)$$

$$\left(\rho_4^E\right)^d = -0.5 - i, \;\; \Sigma_4^d = diag(0.0102, \; 0.0099)$$

where $\left(\rho_3^E\right)^d$ and $\left(\rho_4^E\right)^d$ are the eigenvalues on the damping ratio($\xi = 0.447$) and natural frequency($\omega_n = 1.12$) of the longitudinal short-period mode. The mean of desired modal matrix is selected to correspond with the desired eigenvalues as follows:

$$\left(\Phi^E\right)^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-i & 1+i \\ 0 & 0 & 1+i & 1-i \end{bmatrix}$$

The mean and covariance of the absolute misalignment angle using *Eq.*(10) are as follows.

$$\theta_1^d(p) \sim (0, \; 0) \text{ rad}$$

$$\theta_2^d(p) \sim (0, \; 0) \text{ rad}$$

$$\theta_3^d(p) \sim (-0.0728, \; 0.0045) \text{ rad}$$

$$\theta_4^d(p) \sim (\; 0.0728, \; 0.0045) \text{ rad}$$

According to the design procedure of the proposed algorithm, feedback gain matrix which consists of probabilistic elements can be obtained through the time interval [0, 5000]. The variation of the Frobenius norm $\left(\|K\|_F \equiv \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |k_{ij}|^2} \right)$ of feedback gain matrix is shown in Fig. 2. The mean of feedback gain matrix can be obtained as follows:

$$K^E = \begin{bmatrix} 0.4158 & 200.4464 & 28.3421 & -121.4804 \\ 0.0708 & 20.9026 & -2.4931 & -39.0587 \end{bmatrix}$$

The mean and covariance of the closed-loop eigenvalues can be obtained as follows:

$$\left(\rho_1^E\right)^a = -5.1525, \;\; \sigma^a_{\tilde{\sigma}_1} = 0.6444$$

$$\left(\rho_2^E\right)^a = -4.0005, \;\; \sigma^a_{\tilde{\sigma}_2} = 0.3474$$

$$\left(\rho_3^E\right)^a = -0.4982 - 0.9992i, \;\; \Sigma_3^a = diag(0.0099, \; 0.0101)$$

$$\left(\rho_4^E\right)^a = -0.4982 + 0.9992i, \;\; \Sigma_4^a = diag(0.0099, \; 0.0101)$$

The probability of stability of the closed-loop system is $S = 1$. The mean matrix of the achievable modal matrix, $\left(\Phi^E\right)^a$, can be achieved in least square sense as follows:

$$\begin{bmatrix} 0.541 & 0.8349 & 1 & 1 \\ 0.0608 & 0.1533 & -0.0004 - 0.0000i & -0.0004 + 0.0000i \\ -0.2249 & -0.4141 & -0.0017 - 0.0000i & -0.0017 + 0.0000i \\ 0.0476 & 0.1039 & 0.0013 - 0.0000i & 0.0013 + 0.0000i \end{bmatrix}$$

The mean and covariance of the absolute misalignment angle for closed-loop S-eigenvectors are as follows:

$$\theta_1^a(p) \sim (\;\; -1.0684e-012, \; 7.5822e-021) \text{ rad}$$

$$\theta_2^a(p) \sim (1.4593e-010, \; 1.8232e-015) \text{ rad}$$

$$\theta_3^a(p) \sim (\;\; -1.2212e-006, \; 1.5576e-008) \text{ rad}$$

$$\theta_4^a(p) \sim (\, 1.2212e-006, \; 1.5576e-008) \text{ rad}$$

These result show the perturbation of S-eigenvectors rarely to raise around the mean of achievable eigenvector. The closed-loop eigenvalues distribution and its PDF for the flight control application on two-dimensional complex plane and three-dimensional probability density complex plane are shown in Fig. 3 and 4. All of eigenvalues are located on the left-half plane, and each eigenvalue varies with its mean value.

6. Conclusions

In this paper, first, the S-eigenvalue concept and its corresponding S-eigenvector pair for linear continuous-time systems with probabilistic uncertainty was proposed. The proposed concept is concerned with the perturbation of eigenvalues due to the probabilistic variable parameters in the dynamic model of a plant. Also, S-stability was presented on the appropriate random characteristics of perturbations to

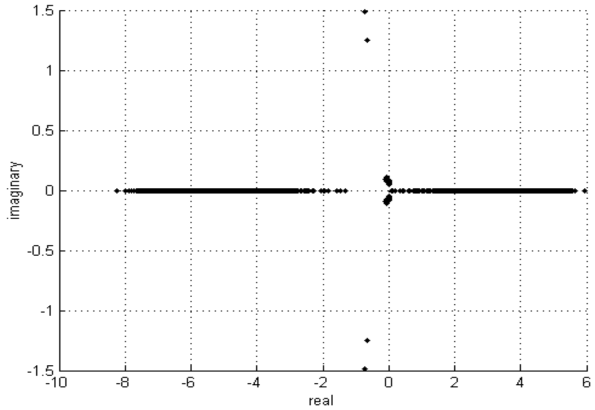

Fig. 1. Open-loop eigenvalues distribution for the flight control application on two-dimensional complex plane

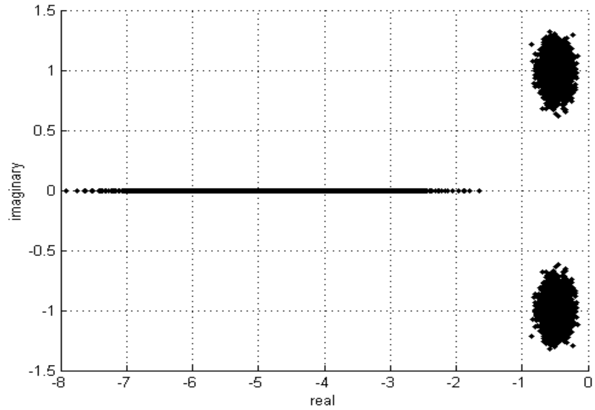

Fig. 3. Closed-loop eigenvalues distribution for the flight control application on two-dimensional complex plane

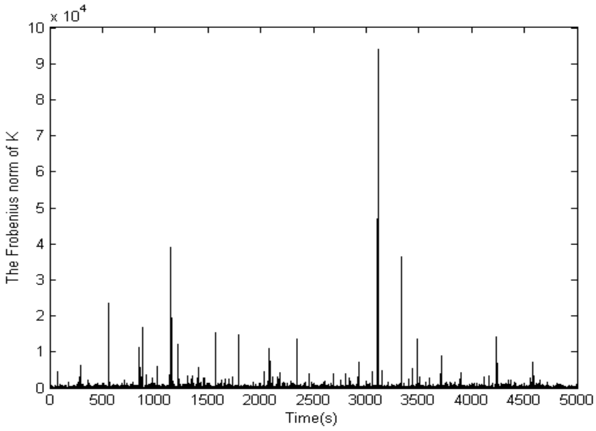

Fig. 2. The variation of feedback gain matrix

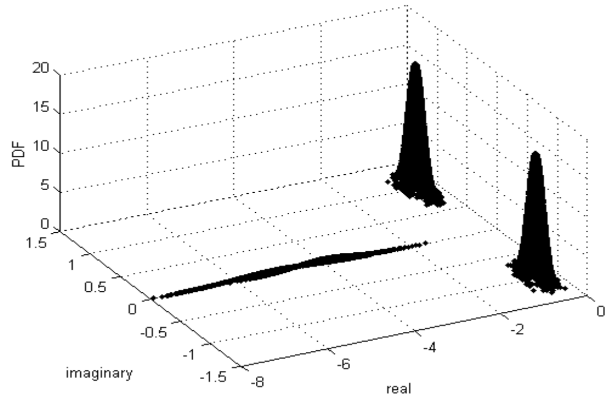

Fig. 4. Closed-loop eigenvalues distribution for the flight control application on three-dimensional probability density complex plane

maintain the proper stability behavior of the overall system. Finally, S-eigenstructure assignment scheme via a Sylvester equation approach based on the S-eigenvalue concept was proposed. The proposed design scheme was applied to the longitudinal dynamics of open-loop-unstable aircraft with possible uncertainties in aerodynamic and thrust effects as well as separate dynamic pressure. These results explicitly characterized how S-eigenvalues in the complex plane may impose stability on the system.

References

[1] Liu, G. P., and Patton, R. J., *Eigenstructure Assignment for Control System Design*, John Wiley & Sons, 1998.

[2] Fahmy, M. M., and Tantaway, H. S., "Eigenstructure Assignment via Linear State-Feedback Control," *International Journal of Control*, Vol. 40, No. 1, pp. 161-178, 1984.

[3] Sobel, K. M., and Shapiro, E. Y., "A Design Methodology for Pitch Pointing Flight Control Systems," *Journal of Guidance, Control and Dynamics*, Vol. 8, No. 2, pp. 181-187, 1985.

[4] Sobel, K. M., and Shapiro, E. Y., "Eigenstructure assignment for design of multimode flight control systems," *IEEE Control Systems Magazine*, pp. 9-15, May 1985.

[5] Lin, C. F., *Advanced Control Systems Design*, Englewood Cliffs, NJ, Prentice Hall, Inc., 1994.

[6] Garg, S., "Robust Eigenspace Assignment Using Singular Value Sensitivities," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, pp. 416-424, 1991.

[7] Horn, R. A., and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, 1985.

[8] Maybeck, P. S., *Stochastic Models, Estimation, and Control, Vol. 1,* Mathematics in Science and Engineering, Vol. 141, Academic Press, pp. 101-111, 1979.

[9] John, S. B., *Fundamentals of Linear State Space Systems,* WCB/ McGraw-Hill, pp. 147-193, 1999.

[10] Stengel, R. F. and Ray, L. R., "Stochastic Robustness of Linear Time-Invariant Control Systems," *IEEE Transactions on Automatic Control*, Vol. 36, No. 1, pp. 82-87, 1991.