Design of First Order Controllers with Time Domain Specifications(**ICCAS 2003**)

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Abstract: This paper considers the problem of determining a set of stabilizing first order controller gains, for a given linear time invariant plant, that meets or exceeds closed loop step response specifications. The method utilizes two recent results: For a given system, (1) finding a set of stabilizing first order controller gains and (2) the relationship between time response (overshoot and speed) and the coefficients of the characteristic polynomial. The method allows us to extract a subset of first order controller gains that meets stability as well as time domain performance requirements. The computations involved are the intersections of two dimensional sets described by linear and quadratic inequalities in the controller design space. It is illustrated by examples.

Keywords: CRA, first order controller, time domain specification

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1. INTRODUCTION

In linear control system design, the majority of practical control systems are based on simple and fixed structure controllers such as PID and first order controllers. And the transient response control is one of the most important requirements. However, there are very few results dealing with design of low order controllers considering these problems. Over the last 40 years, feedback control design problems have been solved within the broad framework of optimal control theory. Unfortunately, results often have unnecessarily high order controllers. Recently, Ho[1] and Tantaris[2,3] explicitly computes all the stabilizing set of PID and first order controller gains. And then they select controller parameters to meet frequency domain specifications. Most of approaches for the time response analysis are related to the pole-zero locations of system. However, the precise relationship between pole locations and transient response is not yet known except for the case of second order system.

In the 1960's, Naslin[4,5] presented a different method that characterizes the transient response in terms of the coefficients of characteristic polynomial rather than its roots by defining the so-called characteristic ratios and pulsatances. He claimed by some empirical observations that these parameters are closely related to the overshoot and the settling time of system step response. Unfortunately, Naslin's idea has hardly been studied due to the advent of state space method proposed by Kalman, but in recent years this led to new classical control design methods, such as the Characteristic Ratio Assignment (CRA) by Kim[6]. CRA design method formulates a model matching problem which reference model is selected by all-pole system having desired characteristic polynomial. Such polynomials are constructed only by the characteristic ratios $\alpha_i s$ and the generalized time constant τ. It was shown by Lipatov[7], Manabe[8] and Kim[6] that $\alpha_i s$ relate to the stability as well as the damping, and that τ controls the speed of step response. Even though a linear controller resulting in good transient response can be designed by CRA, this method is applicable only when all the coefficients of the characteristic polynomial can be freely assigned. This corresponds to the case when a high enough order controller is used so that the total freedom is available to achieve any target characteristic polynomial.

We designed PID controller using CRA with three characteristic ratio parameters, α_1, α_2 and α_3 which were obtained through the relation between frequency response and step response[9,10]. We also analyzed the sensitivity function of n'th order system with respect to the changes of coefficients themselves, and characteristic ratios of denominator polynomial to determine the sensitivity of step response to the change of characteristic ratios[11]. As an important result, it says that, for any fixed τ, the step response is dominantly affected merely by α_1, α_2 and α_3 regardless of the order of denominator higher than 4. This means that the rest of the characteristic ratios have little effect on the step response but are crucial for maintaining stability. This property plays an important role in developing a new first order controller design.

Now suppose that first order controller for a given LTI plant be designed under the time response specifications. Since all characteristic polynomial coefficients are not freely assignable, the model matching problem via CRA is in general unsolvable. Unlike the full order case, it may be difficult to maintain the stability through CRA. In this paper, we circumvent this problem by using all the stabilizing set of first order controller[2]. The proposed method in this paper consists of three steps. For a given linear plant, (1) find a set of all stabilizing first order controller gains, (2) assign appropriate values of α_1, α_2, α_3 and τ that meet the damping requirement and let the rest of the characteristic ratios float, and (3) then extract a subset of first order controller gains from the whole stabilizing set by applying conditions of the step (2), so that the time domain performance requirements are satisfied.

With taking advantage of this result, we present in this paper a new first order controller design method that determines an optimal set of first order controller gains that satisfy time response specifications, namely overshoot and speed of response, and ensures the closed loop stability. The computations involved in are intersections of two-dimensional sets satisfying linear or quadratic inequalities in the controller parameter space and can be conveniently displayed to the designer. Examples are given for the illustration.

2. Preliminaries

In this section, CRA properties and the Tantaris's idea[2] to obtain all stabilizing first order controller gains in the control parameter space which stabilize a fixed LTI plant are briefly reviewed .

2.1 Characteristic Ratios and Time Response

Consider a following characteristic polynomial with real positive coefficients:

$$\delta(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \ \forall a_i > 0. \quad (1)$$

The *characteristic ratios* are defined as:

$$\alpha_1 := \frac{a_1^2}{a_0 a_2},\ \alpha_2 := \frac{a_2^2}{a_1 a_3},\ \cdots,\ \alpha_{n-1} := \frac{a_{n-1}^2}{a_{n-2}a_n} \quad (2)$$

and the *generalized time constant* is defined as

$$\tau := \frac{a_1}{a_0}. \quad (3)$$

It was shown in [6] that τ represents the speed of the response of a system with denominator $\delta(s)$. The coefficients a_i of $\delta(s)$ may also be represented in terms of α_i and τ as follows:

$$a_1 = a_0\tau$$
$$a_i = \frac{a_0\tau^i}{\alpha_{i-1}\alpha_{i-2}^2\alpha_{i-3}^3\cdots\alpha_2^{i-2}\alpha_1^{i-1}},\ i = 2,3,\cdots,n \quad (4)$$

We see that for a given set of values α_is, τ and a_0 the corresponding polynomial $\delta(s)$ is completely determined. We set $\alpha_1 = \alpha$ and call it the *principle characteristic ratio*.

Although the definition of time constant is clear for the case of 1st order systems, the corresponding definition is not known when multiple time constants are presented. Kim[6] showed that the *generalized time constant* is precisely related to speed of response. As an important result, it says that the speed of response of a linear all-pole system) can be controlled, while maintaining the exact shape of response, by adjusting the value of τ if its α_is can be kept the same. We consider two systems which have their generalized time constant τ_1 and τ_2 respectively, having the same characteristic ratios. The step response of each system, $y_1(t)$, $y_2(t)$, has the following relationship[6].

$$y_1(t) = y_2\left(\frac{\tau_1}{\tau_2}\cdot t\right),\ \forall t \geqq 0 \quad (5)$$

Kim also proposed the method to obtain the desired closed loop polynomial as followings.

(i) $\alpha_1 > 2$,

(ii) $$\alpha_k = \frac{\sin(\frac{k\pi}{n}) + \sin(\frac{\pi}{n})}{2\sin(\frac{k\pi}{n})}\cdot\alpha_1,\ \ k = 2,\cdots,n-1. \quad (6)$$

The construction mechanism of eq.(6), the so-called K-polynomial, involves only α_1 which must have a value greater than 2. Kim showed that a desired system composed by (6) is always Hurwitz stable and the frequency magnitude of the system is monotonically decreasing. This means that the overshoot of step response can be adjusted by selecting only α_1. We showed, for fixed τ, the three characteristic ratios, α_1, α_2 and α_3, dominantly affect the step response regardless of the order of denominator higher than 4[9,10,11]. As a result, a criterion to find a set of all stabilizing controller gains satisfying the time domain specification is given by the following inequalities :

$$\alpha_1 \geqq \alpha_1^*,\ \alpha_2 \geqq \alpha_2^*,\ \alpha_3 \geqq \alpha_3^*,\ \tau^- \leqq \tau \leqq \tau^+. \quad (7)$$

Where, α_1^*, α_2^*, α_3^*, $\tau^-$and τ^+ are a target set of parameters that corresponds to a closed loop system satisfying the desired time response specifications such as the maximum percent overshoot and the range of permissible settling time. A value of τ^- is used to limit overshoot in the plant with zeros that increase inevitably the overshoot and τ^+ is used for the settling time.

2.2 Set of Stabilizing System by First Order Controller

Consider a feedback control system where the plant is $P(s) = \frac{N(s)}{D(s)}$ with the following cascade first order controllers.

$$C(s) = \frac{x_1 s + x_2}{s + x_3} \quad (8)$$

The characteristic polynomial of a closed loop system is

$$\delta(s) = D(s)(s + x_3) + N(s)(x_1 s + x_2),$$
$$= a^n s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0. \quad (9)$$

The complex root boundary is given by

$$\delta(jw) = 0,\ w \in (0, +\infty]. \quad (10)$$

and real root boundary is given by

$$\delta(0) = 0,\ \delta_{n+1} = 0. \quad (11)$$

Tantaris[2] shows that the parameter space is separated into disjoint open subsets by eq.(10) and (11). For fixed x_3, these regions can be determined in the corresponding $x_1 - x_2$ plane by solving linear equations for the boundary crossing points as a function of frequency in the closed form. So the root invariant regions can be generated explicitly. It is remarkable that a number of roots that lie at the LHP in each closed region is always the same. From these calculations, the parameter space regions that consists of all stabilizing first order controllers are obtained by sweeping over the parameter x_3.

3. Choosing Stabilizing First Order gains with Time Response Specifications

Consider a feedback control system with first order controller which has the two-parameter controller configuration shown in Fig.1. This configuration has the advantage that the zeros of the controller are not added to the zeros of the overall system, and the configuration is common in practice. Where K_{dc} is the compensator for dc gain of a closed loop system.

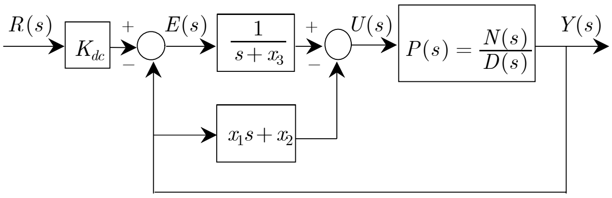


Fig. 1 Two-parameter feedback configuration

The plant transfer function and the control law are

$$P(s) = \frac{n_m s^m + n_{m-1} s^{m-1} + \cdots + n_1 s + n_0}{d_p s^p + n_{p-1} s^{p-1} + \cdots + d_1 s + d_0}, \quad \text{where} \quad m \leqq p, \tag{12}$$

$$U(s) = \frac{1}{s+x_3} E(s) - (x_1 s + x_2) Y(s). \tag{13}$$

The closed-loop characteristic polynomial is given by eq.(9). We compute the characteristic ratios and the generalized time constant:

$$\alpha_1 = \frac{a_1^2}{a_0 a_2} = \frac{(d_0 + d_1 x_3 + n_0 x_1 + n_1 x_2)^2}{(d_0 x_3 + n_0 x_2)(d_1 + d_2 x_3 + n_1 x_1 + n_2 x_2)}, \tag{14}$$

$$\alpha_2 = \frac{a_2^2}{a_1 a_3} = \frac{(d_1 + d_2 x_3 + n_1 x_1 + n_2 x_2)^2}{(d_0 + d_1 x_3 + n_0 x_1 + n_1 x_2)(d_2 + d_3 x_3 + n_2 x_1 + n_3 x_2)} \tag{15}$$

$$\alpha_3 = \frac{a_3^2}{a_2 a_4} = \frac{(d_2 + d_3 x_3 + n_2 x_1 + n_3 x_2)^2}{(d_1 + d_2 x_3 + n_1 x_1 + n_2 x_2)(d_3 + d_4 x_3 + n_3 x_1 + n_4 x_2)} \tag{16}$$

$$\tau = \frac{a_1}{a_0} = \frac{(d_0 + d_1 x_3 + n_0 x_1 + n_1 x_2)}{(d_0 x_3 + n_0 x_2)}. \tag{17}$$

The set of all stabilizing first order gains for given system $P(s)$ is found by a set of linear equations as a function of frequency in closed form in $x_1 - x_2$ space for fixed x_3[2]. To find out the first order controller gains to meet time domain specifications, the regions included eq.(7) are redefined as follows:

$$\begin{aligned} K_1 &:= \{(x_1, x_2, x_3),\ \alpha_1 \geq \alpha_1^*\} \\ K_2 &:= \{(x_1, x_2, x_3),\ \alpha_2 \geq \alpha_2^*\} \\ K_3 &:= \{(x_1, x_2, x_3),\ \alpha_3 \geq \alpha_3^*\} \\ K_\tau &:= \{(x_1, x_2, x_3),\ \tau^- \leqq \tau \leqq \tau+\} \end{aligned} \tag{18}$$

Then the region of interest which satisfies both performance and stability requirements is:

$$K^* := K_1 \cap K_2 \cap K_3 \cap K_\tau \cap K. \tag{19}$$

The inequalities in eq.(7) can be expressed in terms of controller parameters by eq.(14)~(16) for fixed x_3 as follows :

$$f_i(x_1, x_2) := x Q_i x^T + 2 q_i x + r_i \leqq 0 \tag{20}$$

where

$i=1$ for α_1, $i=2$ for α_2, $i=3$ for α_3, and $x = [x_1 .\ x_2]$,

$$Q_{11} = \alpha_i^* n_{i-2} n_i - n_{i-1}^2,$$

$$Q_{12} = Q_{21} = \frac{\alpha_i^*}{2} n_{i-2} n_{i+1} + n_{i-1} n_i \left(\frac{\alpha_i^*}{2} - 1\right),$$

$$Q_{22} = \alpha_i^* n_{i-1} n_{i+1} - n_i^2,$$

$$q_1 = \left(\frac{\alpha_i^*}{2}(n_i d_{i-1} + n_{i-2} d_{i+1}) - d_i n_{i-1}\right) x_3 + \frac{\alpha_i^*}{2}(n_i d_{i-2} + n_{i-2} d_i) - d_{i-1} n_{i-1},$$

$$q_2 = \left(\frac{\alpha_i^*}{2}(n_{i+1} d_{i-1} + n_{i-1} d_{i+1}) - d_i n_i\right) x_3 + \frac{\alpha_i^*}{2}(n_{i+1} d_{i-2} + n_{i-1} d_i) x_3 - d_{i-1} n_i,$$

$$Q_i^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad q_i = [\ q_1 \quad q_2\ ]$$

$$r_i = \alpha_i^* (d_{i-2} d_{i+1} + d_{i-1} d_i + d_{i-1} d_{i+1} x_3) x_3 - \alpha_i^* d_{i-2} d_i - x_3 (2 d_{i-1} d_i + d_i^2 x_3) - d_{i-2}^2$$

Similarly, from eq.(7), the τ condition appears as a linear function of (x_1, x_2, x_3) as :

$$\tau^- \leqq \frac{d_0 + d_1 x_3 + n_0 x_1 + n_1 x_2}{d_0 x_3 + n_0 x_2} \leqq \tau^+ \tag{21}$$

The solution of x_2 in eq.(20) has a linear or quadratic inequality equation form as an order of $P(s)$. The constraints eq.(20) and(21) can be displayed on the (x_1, x_2) plane for a fixed x_3 as shown in Fig. 2. We can see visually that an admissible subset of (x_1, x_2) can be easily determined graphically or alternatively by nonlinear programming methods. By sweeping x_3 over all admissible values in the stabilizing set and repeating the above procedure at every x_3, the complete subset of first order gains achieving specifications is obtained.

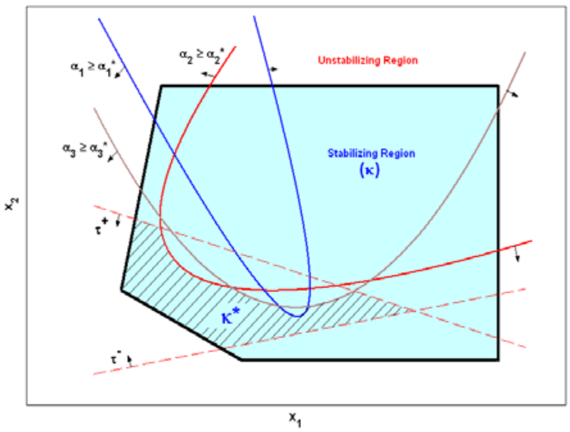


Fig. 2 An admissible set of (x_1, x_2) pairs under a prescribed $\alpha_1{}^*$, $\alpha_2{}^*$, $\alpha_3{}^*$, $\tau^-$and τ^+

4. Illustrative Examples

In this section, we consider two examples to show how the presented method applies to the problem of determining an admissible set of first order controllers which meets a given time domain specification, in particular, the overshoot and the settling time requirements. One example is an all-pole system, while the other plant has minimum phase zero.

Example 1 (All pole system) : Consider the feedback system in Fig. 2 with the following unstable plant with poles at $-2.48 \pm j3.12$, $-4.02 \pm j1.45$, and 1.

$$P(s) = \frac{1}{s^5 + 12s^4 + 61s^3 + 144s^2 + 71s - 290}$$

The objective is to design a set of first order controllers such that the closed loop step responses have about 0.1% overshoot and 1% settling time of 20 secs. Using the algorithm in[2], we obtained the set of all stabilizing first order controller gains with $x_3 \in [-0.3, \infty]$. Fig. 3 shows this region when $x_3 \in [1, 10]$. (x_1, x_2) templates have been depicted at every fixed x_3 with a step size of 1.0. Next, we need to choose $\alpha_1{*}$, $\alpha_2{*}$ and $\alpha_3{*}$ such that the given overshoot specification is satisfied. The following steps describe this selection procedure.

1. **Choose a set of $\alpha_i{'s}$ values using the algorithm in[6]. Alternatively, select $\alpha_1{*}$, $\alpha_2{*}$ and $\alpha_3{*}$ to be any values greater than 2.**
2. **Assign τ**
3. **Plot the step response.**
4. **If overshoot is high, increase $\alpha_1{*}$, $\alpha_2{*}$ and $\alpha_3{*}$.**
5. **If some responses are slow, reduce τ^{+}**

The set of target models $T(s)$ can be represented by

$$T(s) = \frac{K_0 N(s)}{\delta(s)} = \frac{K_0}{a_6 s^6 + a_5 s^5 + \cdots + a_1 s + a_0}$$

where

$$a_0 = -290x_3 + x_2, \quad a_1 = \tau a_0, \quad a_2 = \frac{\tau^2 a_0}{\alpha_1^*},$$

$$a_3 = \frac{\tau^3 a_0}{\alpha_2^* \alpha_1^{*2}}, \quad \cdots, \quad a_6 = \frac{\tau^6 a_0}{\alpha_5^* \alpha_4^{*2} \cdots \alpha_1^{*5}}$$

Using the steps described above, it is very easy to find that $\alpha_1 \geq 2.265$, $\alpha_2 \geq 1.7610$ and $\alpha_3 \geq 1.6365$ meet the damping condition well. Then from the three inequalities eq.(20), we initially applied a feasible range of generalized time constants $\tau^* \in [\tau_{\min}, \tau_{\max}]$=[0.01, 100]. Fig. 4 shows an admissible subset of (x_1, x_2) pairs for fixed x_3*=5.0 Fig. 5(a) shows the subset of adjustable first order controller gains that corresponds to the first candidate set of $(\alpha_1{*}, \alpha_2{*}, \alpha_3{*}, \tau{*})$.

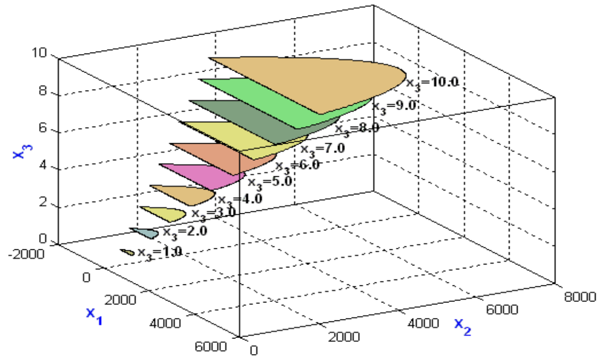


Fig. 3 All stabilizing set of (x_1, x_2, x_3) in Example 1.

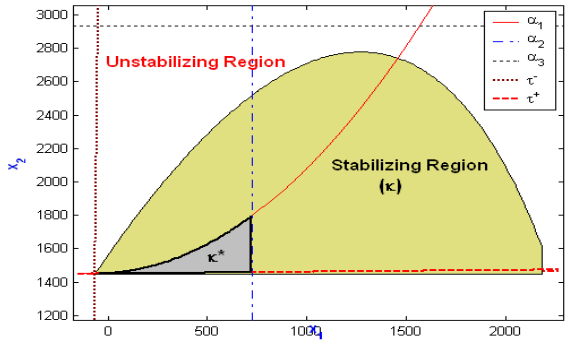
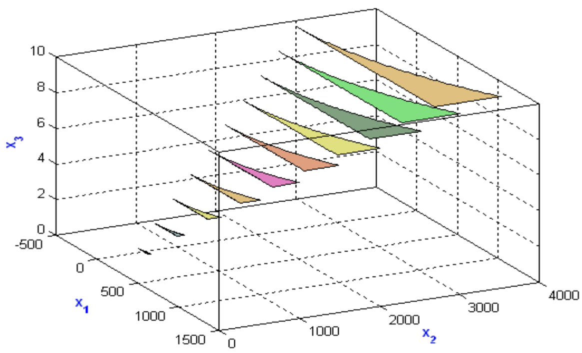
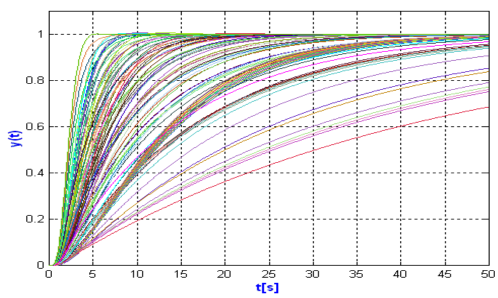


Fig. 4 An admissible set of (x_1, x_2) pairs with x_3*=5.0 in Example 1.



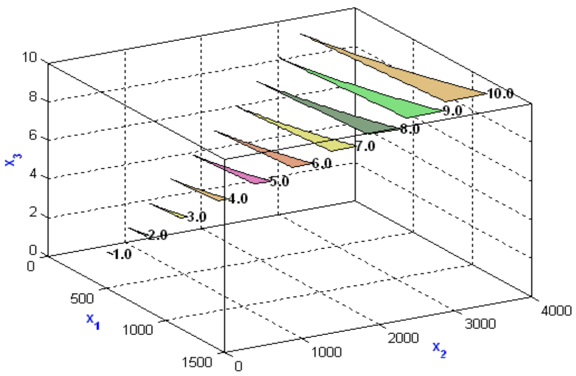
(a) (x_1, x_2, x_3) set



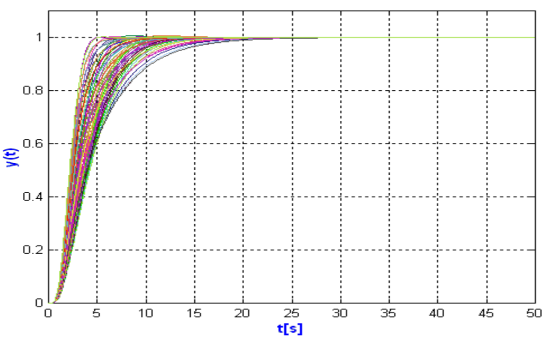
(b) Step responses

Fig. 5 An admissible first order controller parameter set and its step responses with the specification of overshoot in Example 1.

Fig. 5(b) shows the step responses of this set. As seen in the figure, the maximum overshoot meets the given specification but the settling time of the response is delayed because we have not yet carried out the procedure to choose a proper range of τ. The delayed response problem may be easily fixed by reducing the maximum range of generalized time constants. Thus, we tailored the range of generalized time constants to lie in $\tau^* \in [\tau^-, \tau^+] = [0.01, 5.8]$. The corresponding set of controllers and step responses are shown in Fig. 6. It is seen that these sets meet the given performance requirements well.



(a) (x_1, x_2, x_3) set



(b) Step responses

Fig. 6 An admissible first order controller parameter set and its step responses with the time domain specification in Example 1.

Example 2 (Plant with minimum phase zero) : Consider the same feedback system used in example 1. And let add minimum phase zero at -1.0 to that plant.

$$P(s) = \frac{s+1}{s^5 + 12s^4 + 61s^3 + 144s^2 + 71s - 290}$$

The design specifications are to meet the step response maximum overshoot of 1% and the 1% settling time of 20 secs. Using the algorithm in [2] again, we obtained the stabilizing first order controller set in which the stable region of (x_1, x_2) exists only for $x_3 \in [-1.0, \infty]$. Fig. 7 shows this region when $x_3 \in [0.7, 1.6]$. In this case, the closed-loop reference model can be described by a 5th-order model including a fixed real zero.

$$T(s) = \frac{K_0 N(s)}{\delta(s)} = \frac{K_0(s+1)}{a_6 s^6 + a_5 s^5 + \cdots + a_1 s + a_0}$$

As in example 1, we determined the specifications $\alpha_1{}^* = 2.265$, $\alpha_2{}^* = 1.7610$ and $\alpha_3{}^* = 1.6365$, and $\tau^* \in [5.2, 5.7]$ through some experiments. The final subset of first order controllers with satisfactory performances is shown in Fig. 8 and step responses of this set are in Fig. 9.

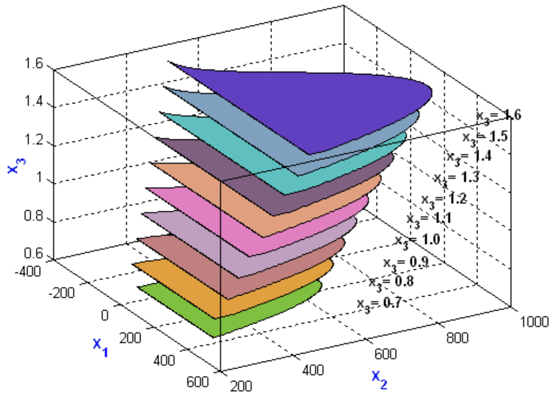


Fig. 7 All stabilizing set of (x_1, x_2, x_3) in Example 2.

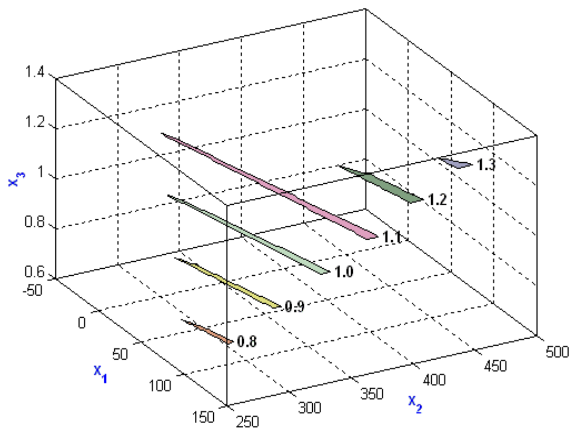


Fig. 8 First order controller gains set with satisfactory specifications in Example 2..

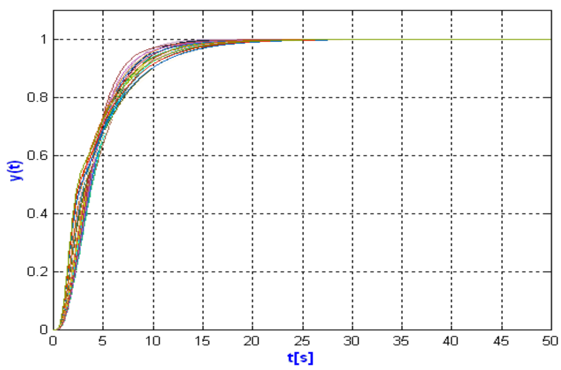


Fig. 9 Step responses corresponding to parameter set of Fig. 8

5. Conclusions

In this paper we have extended earlier results on time response control design to the case where the controller is of first order type. Exploiting the recent result on finding the set of all stabilizing first order controller gains for a given LTI plant, we propose a design scheme based on characteristic ratio and generalized time constant assignment as an approach toward transient response design. As a result, it is important to note that the computations of this algorithm involved in are intersections of two dimensional sets satisfying linear or quadratic inequalities in the controller parameter spaces and can be conveniently displayed to the designer. The graphical display of sets of feasible solutions using 2-D or 3D graphics is an attractive feature of the design method and would allow the imposition of further design requirements. We convince ourselves that the proposed first order controller design method is valuable in the practical sense, since it guarantees that any of the overshoot and the settling time specifications can be satisfied almost completely.

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