

## Bayesian Test for Equality of Coefficients of Variation in the Normal Distributions

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### Abstract

When  $X$  and  $Y$  have independent normal distributions, we develop a Bayesian testing procedure for the equality of two coefficients of variation. Under the reference prior of the coefficient of variation, we propose a Bayesian test procedure for the equality of two coefficients of variation using fractional Bayes factor. A real data example is provided.

*Keywords* : Fractional Bayes Factor; Reference Prior; Normal Distribution; Coefficients of Variations.

### 1. INTRODUCTION

The coefficient of variation is an important parameter in many physical, biological and medical sciences. In general, it measures the consistency or uniformity of a set of observations on a random variable. Since the coefficient of variation is the standard deviation per unit mean, it represents a measure of relative variability. Groups can have the same relative variability even if the means and variances of the variable of interest are different.

The present paper focuses on Bayesian testing procedure for the equality of two coefficients of variation. In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' priors or reference priors (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the

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fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction  $b$ . These approaches have shown to be quite useful in many statistical areas.

For testing the equality of coefficients of variation (CV), Miller and Karson (1977) presented a test for the equality of two CVs. Doornbos and Dijkstra (1983) developed a likelihood ratio test and a non central  $t$  test for the case of  $k$  normal samples of possible unequal sizes. The likelihood ratio test involves an algebraically unsolvable equation when more than two populations are considered. So Gupta and Ma (1996) provided a better method of solving this equation numerically than the one suggested by Doornbos and Dijkstra (1983) and developed a new test, the so called score test. Rao and Vidya (1992) provided a Wald test for testing the equality of CVs in two populations with equal sample sizes.

Almost all the work mentioned above is the analysis based on the classical point of view, there is a little work on this problem from the viewpoint of Bayesian framework. And we feel a strong necessity to develop objective Bayesian procedure for dealing this problem. So we want to develop the Bayesian test procedure for the equality of two coefficients of variation using Bayes factor. Using the noninformative priors developed previously, we calculate the posterior probabilities of the hypotheses using the fractional Bayes factor of O'Hagan (1995). Our testing for the equality of coefficients of variation will imply that the two means are of equal sign. Thus we can, without loss of generality, assume that the two means are positive (Sinha, Rao and Clement, 1978; Gupta, Ramakrishnan and Zhou, 1999).

The outline of the remaining sections is as follows. In Section 2, using the reference priors, we provide the Bayesian testing procedure based on the fractional Bayes factor for the testing equality of two coefficients of variation. In Section 3, a real example is given.

## 2. BAYESIAN TEST USING THE FRACTIONAL BAYES FACTOR

### 2.1 Preliminaries

Models (or Hypotheses)  $H_1, H_2, \dots, H_q$  are under consideration, with the data  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  having probability density function  $f_i(\mathbf{x} | \theta_i)$  under model  $H_i, i=1, 2, \dots, q$ . The parameter vectors  $\theta_i$  are unknown. Let  $\pi_i(\theta_i)$  be the prior distribution of model  $H_i$ , and let  $p_i$  be the prior probabilities of model  $H_i, i=1, 2, \dots, q$ . Then the posterior probability that the model  $H_i$  is true is

$$P(H_i | \mathbf{x}) = \left( \sum_{j=1}^q \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \quad (1)$$

where  $B_{ji}$  is the Bayes factor of model  $H_j$  to model  $H_i$  defined by

$$B_{ji} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})} = \frac{\int f_j(\mathbf{x} | \boldsymbol{\theta}_j) \pi_j(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}. \quad (2)$$

The  $B_{ji}$  interpreted as the comparative support of the data for the model  $j$  to  $i$ . The computation of  $B_{ji}$  needs specification of the prior distribution  $\pi_i(\boldsymbol{\theta}_i)$  and  $\pi_j(\boldsymbol{\theta}_j)$ . Usually, one can use the noninformative prior, often improper, such as uniform prior, Jeffreys prior, reference prior or probability matching prior. Denote it as  $\pi_i^N$ . The use of improper priors  $\pi_i^N(\cdot)$  in (2) causes the  $B_{ji}$  to contain unspecified constants. To solve this problem, O'Hagan (1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow.

When the  $\pi_i^N(\boldsymbol{\theta}_i)$  is noninformative prior under  $H_i$ , equation (2) becomes

$$B_{ji}^N = \frac{\int f_j(\mathbf{x} | \boldsymbol{\theta}_j) \pi_j^N(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}.$$

Then the fraction Bayes factor (FBF) of model  $H_j$  versus model  $H_i$  is

$$B_{ji}^F = \frac{q_j(b, \mathbf{x})}{q_i(b, \mathbf{x})},$$

where

$$q_i(b, \mathbf{x}) = \frac{\int f_i(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i}{\int f_i^b(\mathbf{x} | \boldsymbol{\theta}_i) \pi_i^N(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i},$$

and  $f_i(\mathbf{x} | \boldsymbol{\theta}_i)$  is the likelihood function and  $b$  specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction  $b$ . One frequently suggested choice is  $b = m/n$ , where  $m$  is the size of the minimal training sample, assuming this is well defined. (see O'Hagan, 1995 and the discussion by Berger and Mortera of O'Hagan, 1995).

## 2.2 Bayesian Test

Suppose that  $\mathbf{X}=(X_1, \dots, X_{n_1})$  is a random sample of size  $n_1$  from a normal population with mean  $\mu_1$  and variance  $\mu_1^2\gamma_1^2$  and  $\mathbf{Y}=(Y_1, \dots, Y_{n_2})$  is a random sample of size  $n_2$  from a normal population with mean  $\mu_2$  and variance  $\mu_2^2\gamma_2^2$ . Here  $\gamma_1$  and  $\gamma_2$  are the coefficient of variation for each population. Then the joint probability density function is

$$f(\mathbf{x}, \mathbf{y} \mid \mu_1, \mu_2, \gamma_1, \gamma_2) = (2\pi)^{-(n_1+n_2)/2} \gamma_1^{-n_1} \gamma_2^{-n_2} \mu_1^{-n_1} \mu_2^{-n_2} \\ \times \exp\left\{-\frac{1}{2\gamma_1^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{1}{2\gamma_2^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\},$$

where  $\mu_1 > 0$ ,  $\mu_2 > 0$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$ .

We want to test the hypotheses  $H_1: \gamma_1 = \gamma_2$  vs.  $H_2: \gamma_1 \neq \gamma_2$ . The hypothesis  $H_1$  indicate the common coefficient of variation. Our interest is to develop a Bayesian test for  $H_1$  vs.  $H_2$  which is an alternative to the classical tests.

Under the hypothesis  $H_1$ , one-at-a-time reference prior for  $\gamma (\equiv \gamma_1 = \gamma_2)$ ,  $\mu_1$  and  $\mu_2$  is

$$\pi_{H_1}(\gamma, \mu_1, \mu_2) = \mu_1^{-1} \mu_2^{-1} \gamma^{-1} (1 + 2\gamma^2)^{-1/2}, \quad \mu_1, \mu_2, \gamma > 0.$$

This reference prior developed by Lee and Kang (2003). Also they proved that the posterior density under this reference prior is proper. The likelihood function under  $H_1$  is

$$L(\gamma, \mu_1, \mu_2 \mid \mathbf{x}, \mathbf{y}) = (2\pi)^{-(n_1+n_2)/2} \mu_1^{-n_1} \mu_2^{-n_2} \gamma^{-(n_1+n_2)} \\ \times \exp\left\{-\frac{1}{2\gamma^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{1}{2\gamma^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\}.$$

Then the element of FBF under  $H_1$  is given by

$$\int_0^\infty \int_0^\infty \int_0^\infty L^b(\gamma, \mu_1, \mu_2 \mid \mathbf{x}, \mathbf{y}) \pi_1(\gamma, \mu_1, \mu_2) d\gamma d\mu_1 d\mu_2 \\ = (2\pi)^{-b(n_1+n_2)/2} \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-bn_1-1} \mu_2^{-bn_2-1} \gamma^{-b(n_1+n_2)-1} (1 + 2\gamma^2)^{-\frac{1}{2}} \\ \times \exp\left\{-\frac{b}{2\gamma^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{b}{2\gamma^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\gamma d\mu_1 d\mu_2.$$

Let

$$S(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-n_1-1} \mu_2^{-n_2-1} \gamma^{-(n_1+n_2)-1} (1 + 2\gamma^2)^{-1/2} \\ \times \exp\left\{-\frac{1}{2\gamma^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{1}{2\gamma^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\gamma d\mu_1 d\mu_2$$

and

$$S^b(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-bn_1-1} \mu_2^{-bn_2-1} \gamma^{-b(n_1+n_2)-1} (1+2\gamma^2)^{-1/2} \\ \times \exp\left\{-\frac{b}{2\gamma^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{b}{2\gamma^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\gamma d\mu_1 d\mu_2.$$

Then

$$q_1(b, \mathbf{x}, \mathbf{y}) = \frac{(2\pi)^{-\frac{(n_1+n_2)}{2}} S(\mathbf{x}, \mathbf{y})}{(2\pi)^{-\frac{b(n_1+n_2)}{2}} S^b(\mathbf{x}, \mathbf{y})}.$$

For the  $H_2$ , one-at-a-time reference prior for  $\mu_1, \mu_2, \gamma_1$  and  $\gamma_2$  is

$$\pi_{H_2}(\mu_1, \mu_2, \gamma_1, \gamma_2) = \pi(\mu_1, \gamma_1)\pi(\mu_2, \gamma_2) \\ = \mu_1^{-1}\mu_2^{-1}\gamma_1^{-1}\gamma_2^{-1}(1+2\gamma_1^2)^{-1/2}(1+2\gamma_2^2)^{-1/2}, \quad \mu_1, \mu_2, \gamma_1, \gamma_2 > 0.$$

Note that the propriety of the posterior distribution under this reference prior given in Appendix 1. The likelihood function under  $H_2$  is

$$L(\mu_1, \mu_2, \gamma_1, \gamma_2 | \mathbf{x}, \mathbf{y}) = (2\pi)^{-(n_1+n_2)/2} \mu_1^{-n_1} \mu_2^{-n_2} \gamma_1^{-n_1} \gamma_2^{-n_2} \\ \times \exp\left\{-\frac{1}{2\gamma_1^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{1}{2\gamma_2^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\}.$$

Thus the element of FBF under  $H_2$  gives as follows.

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L^b(\mu_1, \mu_2, \gamma_1, \gamma_2 | \mathbf{x}, \mathbf{y}) \pi_2(\mu_1, \mu_2, \gamma_1, \gamma_2) d\mu_1 d\mu_2 d\gamma_1 d\gamma_2 \\ = (2\pi)^{-\frac{b(n_1+n_2)}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-bn_1-1} \mu_2^{-bn_2-1} \gamma_1^{-bn_1-1} \gamma_2^{-bn_2-1} (1+2\gamma_1^2)^{-1/2} (1+2\gamma_2^2)^{-1/2} \\ \times \exp\left\{-\frac{b}{2\gamma_1^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{b}{2\gamma_2^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\mu_1 d\mu_2 d\gamma_1 d\gamma_2.$$

Put

$$T(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-n_1-1} \mu_2^{-n_2-1} \gamma_1^{-n_1-1} \gamma_2^{-n_2-1} (1+2\gamma_1^2)^{-\frac{1}{2}} (1+2\gamma_2^2)^{-\frac{1}{2}} \\ \times \exp\left\{-\frac{1}{2\gamma_1^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{1}{2\gamma_2^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\mu_1 d\mu_2 d\gamma_1 d\gamma_2$$

and

$$T^b(\mathbf{x}, \mathbf{y}) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mu_1^{-bn_1-1} \mu_2^{-bn_2-1} \gamma_1^{-bn_1-1} \gamma_2^{-bn_2-1} (1+2\gamma_1^2)^{-\frac{1}{2}} (1+2\gamma_2^2)^{-\frac{1}{2}} \\ \times \exp\left\{-\frac{b}{2\gamma_1^2\mu_1^2} \sum_{i=1}^{n_1} (x_i - \mu_1)^2 - \frac{b}{2\gamma_2^2\mu_2^2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right\} d\mu_1 d\mu_2 d\gamma_1 d\gamma_2.$$

Then

$$q_2(b, \mathbf{x}, \mathbf{y}) = \frac{(2\pi)^{-\frac{(n_1+n_2)}{2}} T(\mathbf{x}, \mathbf{y})}{(2\pi)^{-\frac{b(n_1+n_2)}{2}} T^b(\mathbf{x}, \mathbf{y})}.$$

Therefore the FBF of  $H_2$  versus  $H_1$  is given by

$$B_{21}^F(\mathbf{x}, \mathbf{y}) = \frac{T(\mathbf{x}, \mathbf{y})S^b(\mathbf{x}, \mathbf{y})}{T^b(\mathbf{x}, \mathbf{y})S(\mathbf{x}, \mathbf{y})}.$$

Note that the element of FBF under  $H_1$  requires a three dimensional integration and the element of FBF under  $H_2$  requires a two dimensional integration. Therefore we have the value of the FBF of  $H_2$  versus  $H_1$ . In Section 3, we investigate our testing procedure and the classical test statistic.

### 3. AN EXAMPLE

The data in Table 1 are taken from Nelson (1990) and represent the hours to failure of 20 motorettes with a new class-H insulation run at 240 °C and 220 °C. It has been observed by Nelson (1990) that lognormal distribution adequately fits at the two temperatures. Note that  $X$  and  $Y$  denote the nature logarithm of the failure times in the Table 1. We thus assume that the below data come from independent normal distributions.

Table 1: Failure Times at the Two Temperatures

$X$ (240 °C)	7.0690	7.0690	7.3271	7.3582	7.3883
	7.4176	7.4176	7.4460	7.4736	7.5771
$Y$ (220 °C)	7.4753	7.7981	7.7981	7.7981	7.7981
	7.7981	8.0417	8.0417	8.0417	8.0417

For the equality of the coefficients of variation, the score test developed by Gupta and Ma (1996). That is, under  $H_1$ , the test statistics is given by

$$Z = \frac{\hat{\gamma}^2(1+2\hat{\gamma}^2)}{2} \left( \frac{a_1^2}{n_1} + \frac{a_2^2}{n_2} \right),$$

where

$$a_1 = \sum_{i=1}^{n_1} \frac{(x_i - \hat{\mu}_1)^2}{\hat{\mu}_1^2 \hat{\gamma}^3} - \frac{n_1}{\hat{\gamma}}, a_2 = \sum_{i=1}^{n_2} \frac{(y_i - \hat{\mu}_2)^2}{\hat{\mu}_2^2 \hat{\gamma}^3} - \frac{n_2}{\hat{\gamma}}.$$

and  $\hat{\mu}_1, \hat{\mu}_2, \hat{\gamma}$  is maximum likelihood estimator of  $\mu_1, \mu_2, \gamma$ . Under  $H_1$ ,  $Z$  has a chi-square distribution with 1 degree of freedom. Since  $\hat{\mu}_1 = 7.354238$ ,  $\hat{\mu}_2 = 7.863381$  and  $\hat{\gamma} = 0.021644$ , the observed value of the test statistic  $Z$  is 0.0113. Hence  $H_1$  not rejected and the  $p$ -value of the test is almost one (Gupta, Ramakrishnan and Zhou, 1999).

The value of fractional Bayes factor of  $H_2$  versus  $H_1$  is  $B_{21}^F = 0.2939$ . We assume that the prior probabilities are equal. Then the posterior probability for  $H_1$  is 0.7729. Thus there are strong evidence for  $H_1$  in terms of the posterior probability.

Therefore both of the classical method and Bayes factor give reasonable answers in this example.

#### APPENDIX 1. Propriety of Posterior Distribution

Under the reference prior  $\pi(\mu, \gamma) = \mu^{-1} \gamma^{-1} (1 + 2\gamma^2)^{-1/2}$ , the joint posterior for  $\mu, \gamma$  given  $\mathbf{x}$  is

$$\pi(\mu, \gamma | \mathbf{x}) \propto \mu^{-(n+1)} \gamma^{-(n+1)} (1 + 2\gamma^2)^{-1/2} \exp\left\{-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\gamma^2 \mu^2}\right\}.$$

Since  $(1 + 2\gamma^2)^{-1/2} \leq 2^{-1/2} \gamma^{-1}$ , thus

$$\pi(\mu, \gamma | \mathbf{x}) \leq C_1 \mu^{-(n+1)} \gamma^{-(n+2)} \exp\left\{-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\gamma^2 \mu^2}\right\}, \tag{3}$$

where  $C_1$  is a constant. Integrating with respect to  $\gamma$  in (3), then

$$C_2 \left[ \sum_{i=1}^n (x_i - \mu)^2 \right]^{-\frac{n+1}{2}} = C_3 \left[ 1 + \frac{n(\bar{x} - \mu)^2}{S^2} \right]^{-\frac{n+1}{2}},$$

where  $S^2 = \sum_{i=1}^n (x_i - \bar{x})^2$  and  $C_2$  and  $C_3$  are a constant. Thus the above form has a Student- $t$  with  $n$  degrees of freedom. Thus the posterior distribution is proper. This completes the proof.  $\square$

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