

## Intrinsic Priors for Testing Two Lognormal Means with the Fractional Bayes Factor

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### ABSTRACT

The Bayes factors with improper noninformative priors are defined only up to arbitrary constants. So, it is known that Bayes factors are not well defined due to this arbitrariness in Bayesian hypothesis testing and model selections. The intrinsic Bayes factor by Berger and Pericchi (1996) and the fractional Bayes factor by O'Hagan (1995) have been used to overcome this problems. This paper suggests intrinsic priors for testing the equality of two lognormal means, whose Bayes factors are asymptotically equivalent to the corresponding fractional Bayes factors. Using proposed intrinsic priors, we demonstrate our results with a simulated dataset.

*Keywords* : Bayes factor, Fractional Bayes factor, Intrinsic prior, Noninformative prior, Reference prior

### 1. Introduction

It has been well known that Bayes factors with proper priors have been very successful in testing or model selection problems. However, in Bayesian analysis, limited informations and time restrictions often force to the use of noninformative priors such as Jeffrey's priors or reference priors. These noninformative priors are usually improper density and the Bayes factors under improper priors are not well defined because these priors are defined only up to arbitrary constants.

Suppose that  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  are random samples from population with a probability density  $f(\mathbf{x}|\theta_i)$ , where  $\theta_i$  is a vector of unknown parameters,  $i=1,2$ . Let  $\Theta_i$  be the parameter space for  $\theta_i$  and  $\pi_i^N(\theta_i)$  be the improper prior density,  $i=1,2$ . The Bayes factor  $B_{21}^N$  of model  $M_2$  to model  $M_1$  is defined by

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$$B_{21}^N = \frac{m_2^N(\mathbf{x})}{m_1^N(\mathbf{x})} = \frac{\int_{\theta_2} f(\mathbf{x} | \theta_2) \pi_2^N(\theta_2) d\theta_2}{\int_{\theta_1} f(\mathbf{x} | \theta_1) \pi_1^N(\theta_1) d\theta_1}, \quad (1.1)$$

where  $m_1^N(\mathbf{x})$  and  $m_2^N(\mathbf{x})$  are the marginal densities under model  $M_1$  and model  $M_2$ , respectively. Since  $\pi_1^N(\theta_1)$  and  $\pi_2^N(\theta_2)$  are improper, the Bayes factor in (1.1) contains arbitrary constants, say  $c_1$  and  $c_2$ . So, the resulting Bayes factor is not well defined.

Several authors including Geisser and Eddy (1979), Spiegelhalter and Smith (1982), and San Martini and Spezzaferri (1984) have made efforts to overcome this problems. O'Hagan (1995) introduced the fractional Bayes factor (FBF) and Berger and Pericchi (1996) suggested the intrinsic Bayes factor (IBF) as model selection criterion. Two methods have been often served as default Bayes factors.

The IBF is the method for removing the arbitrariness by a subset of data called a training sample. There are several papers using IBF (cf Kim (2000); Kim and Sun (2000); Kim, Kang and Kim (2000)). But, IBF approach has some difficulties due to its considerable computational expense for large sample sizes and instability in small sample. Another useful criterion, FBF, is the method for removing the arbitrariness by a portion of the likelihood, which is computed by exponentiating the likelihood to a power  $b$ , where  $0 \leq b \leq 1$ . It is well defined as in the IBF method. Moreover, it does not require a heavy computation and thus much more effective in the sense of computation. Kim and Kim (2000) proposed a Bayesian testing for the comparison of two exponential means using proper intrinsic priors, whose Bayes factors are asymptotically equivalent to the corresponding FBF's. Bae, Kim and Kim (2000) derived proper intrinsic priors to test the equality of two independent normal means with unknown variance.

## 2. Preliminaries

It has known that the Bayes factor  $B_{21}^N$  in (1.1) involves arbitrary constants. Recently two methods for removing this arbitrariness have been proposed, which are to use a subset of data, a training sample, and a portion of the likelihood, a fraction  $b$ .

The intrinsic Bayes factor introduced by Berger and Pericchi (1996) is to use the part of the data as the minimal training sample. A training sample,  $x(l)$  is called a minimal training sample if it has the minimal sample size to guarantee  $0 < m_i^N(x(l)) < \infty$  for all model  $M_i$ ,  $i = 1, 2$ . O'Hagan (1995) proposed the fractional Bayes factor for removing the arbitrariness in (1.1) by using a fractional part of the entire likelihood with the fraction  $b$  instead of training

sample, and suggested the choice of the fraction.

**Definition 1.** The fractional Bayes factor (FBF) of model  $M_2$  to model  $M_1$  is

$$B_{21}^F = B_{21}^N \times CF_{12}(b), \quad (2.1)$$

where the correction factor  $CF_{12}(b)$  is defined as

$$CF_{12}(b) = \frac{\int_{\theta_1} L_1^b(\theta_1) \pi_1^N(\theta_1) d\theta_1}{\int_{\theta_2} L_2^b(\theta_2) \pi_2^N(\theta_2) d\theta_2}.$$

Here,  $L_i(\theta_i)$  is the likelihood function under model  $M_i$ ,  $i=1,2$  and  $b$  is the fraction of the likelihood,  $0 \leq b \leq 1$ . A commonly suggested choice is  $b = m/n$ , where  $m$  is the size of the minimal training sample and  $n$  is the sample size. We will use this choice in our problem.

It is of interest to find proper priors, often called intrinsic priors, so that the Bayes factors using proper priors will be asymptotically equivalent to the corresponding FBF's. This issue was suggested by Berger and Pericchi (1996). To derive the intrinsic priors, we need the following conditions.

**Condition.** As the sample size grows to infinity, the following holds:

1. Under model  $M_1$ ,  $\hat{\theta}_1 \rightarrow \theta_1$ ,  $\hat{\theta}_2 \rightarrow \phi_2(\theta_1)$ ,
2. Under model  $M_2$ ,  $\hat{\theta}_2 \rightarrow \theta_2$ ,  $\hat{\theta}_1 \rightarrow \phi_1(\theta_2)$ ,

where  $\hat{\theta}_1$  and  $\hat{\theta}_2$  are the maximum likelihood estimators (MLE's) under model  $M_1$  and model  $M_2$ . Then, a set of intrinsic priors denoted by  $(\pi_1^I(\theta_1), \pi_2^I(\theta_2))$  is a solution of the following equations:

$$\begin{cases} \frac{\pi_2^I(\phi_2(\theta_1)) \pi_1^N(\theta_1)}{\pi_2^N(\phi_2(\theta_1)) \pi_1^I(\theta_1)} = B_1^*(\theta_1), \\ \frac{\pi_2^I(\theta_2) \pi_1^N(\phi_1(\theta_2))}{\pi_2^N(\theta_2) \pi_1^I(\phi_1(\theta_2))} = B_2^*(\theta_2), \end{cases} \quad (2.2)$$

where for  $i=1,2$ ,  $B_i^*(\theta_i) = \lim_{N \rightarrow \infty} CF_{12}(b)$  under model  $M_i$ .

The noninformative priors  $\pi_1^N(\theta_1)$  and  $\pi_2^N(\theta_2)$  are called starting priors. The solutions of the above equations are not necessarily unique, nor proper. We will find proper priors for given starting priors. Once we derive proper intrinsic priors, the fractional Bayes factor  $B_{21}^F$

can be replaced by the ordinary Bayes factor  $B_{21}^I$  computed based on intrinsic priors.

### 3. Testing two lognormal means

The lognormal distribution has become increasingly widespread in life testing, reliability and other related fields of application, and the lognormal probability density function is given by

$$f(x | \mu, \eta) = \frac{1}{\sqrt{2\pi\eta x}} \exp\left\{-\frac{1}{2\eta} (\log x - \mu)^2\right\}, \quad 0 < x < \infty, \quad -\infty < \mu < \infty, \quad 0 < \eta < \infty,$$

where  $\mu = E(\log X)$  and  $\eta = \text{Var}(\log X)$ . The lognormal distribution will be denoted as the  $LN(\mu, \eta)$ .

Suppose that we have independent random samples  $X_{ij} \sim LN(\mu_i, \eta)$ ,  $i=1, 2$ ,  $j=1, 2, \dots, n_i$ , where  $\eta$  is common and unknown. We are interested in testing the equality of two lognormal means, that is,

$$M_1 : \exp(\mu_1 + \frac{\eta}{2}) = \exp(\mu_2 + \frac{\eta}{2}) \quad v.s. \quad M_2 : \exp(\mu_1 + \frac{\eta}{2}) \neq \exp(\mu_2 + \frac{\eta}{2}).$$

But, because the scale parameters  $\eta$  of two populations are common, it equals to test the equality of two parameters  $\mu_1$  and  $\mu_2$ , that is,

$$M_1 : \mu_1 = \mu_2 (= \mu) \quad v.s. \quad M_2 : \mu_1 \neq \mu_2.$$

Let for  $\theta_1 = (\mu, \eta)$  and  $\theta_2 = (\mu_1, \mu_2, \eta)$ ,  $\Theta_1 = \{(\mu, \eta) | -\infty < \mu < \infty, 0 < \eta < \infty\}$  and  $\Theta_2 = \{(\mu_1, \mu_2, \eta) | -\infty < \mu_1, \mu_2 < \infty, 0 < \eta < \infty\}$ . Let  $Y_{ij} = \log X_{ij}$  and  $T_i = \prod_{j=1}^{n_i} X_{ij}$  for  $j=1, 2, \dots, n_i$ ,  $i=1, 2$ . Let  $N = n_1 + n_2$  with  $n_1/N \rightarrow p$  as  $N \rightarrow \infty$ .

From now on, we will find a set of intrinsic priors  $(\pi_1^I(\theta_1), \pi_2^I(\theta_2))$ , which are proper over  $\theta_i \in \Theta_i$ ,  $i=1, 2$ . We start with the reference priors for model  $M_1$  and model  $M_2$  given by as follows:

$$\pi_1^N(\mu, \eta) = \pi_2^N(\mu_1, \mu_2, \eta) = \frac{1}{\eta}, \quad 0 < \eta < \infty.$$

Then the Bayes factor under the reference priors  $\pi_1^N$  and  $\pi_2^N$  is given by

$$B_{21}^N = \frac{\sqrt{\pi} \sqrt{N}}{\sqrt{n_1 n_2}} \frac{\Gamma((N-2)/2)}{\Gamma((N-1)/2)} \frac{S^{N-1}}{S_{12}^{N-2}}, \quad (3.1)$$

$$\text{where } S^2 = \sum_{i=1}^2 \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y})^2, \quad S_{12}^2 = \sum_{i=1}^2 \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2,$$

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^2 \sum_{j=1}^{n_i} Y_{ij}, \quad \bar{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}, \quad i=1, 2.$$

**Remark.** Note that if we take one observation from each population as a training sample, the marginal density  $m_1^N(x(l))$  is finite, but  $m_2^N(x(l))$  is not finite. So in our situation, the size of the minimal training sample is  $m=3$ .

The correction factor at  $b=3/N$  in (2.1) is given by

$$CF_{12}\left(\frac{3}{N}\right) = \frac{1}{\pi} \frac{\sqrt{n_1 n_2}}{N} \frac{\sqrt{S_{12}^2/N}}{S^2/N}, \quad (3.2)$$

where  $S^2$  and  $S_{12}^2$  are in (3.1).

To find proper priors, we first find the asymptotic behavior of MLE's of  $\hat{\theta}_1 = (\bar{Y}, S^2/N)$  and  $\hat{\theta}_2 = (\bar{Y}_1, \bar{Y}_2, S_{12}^2/N)$  under model  $M_1$  and  $M_2$ .

**Proposition 1.** As  $N \rightarrow \infty$ , under model  $M_1 : \mu_1 = \mu_2 = \mu$ ,

$$\hat{\theta}_1 = (\bar{Y}, S^2/N) \rightarrow \theta_1 = (\mu, \eta) \text{ and } \hat{\theta}_2 = (\bar{Y}_1, \bar{Y}_2, S_{12}^2/N) \rightarrow (\mu, \mu, \eta) \equiv \phi_2(\theta_1).$$

Under model  $M_2 : \mu_1 \neq \mu_2$ ,

$$\hat{\theta}_1 = (\bar{Y}, S^2/N) \rightarrow (p\mu_1 + q\mu_2, \eta + pq(\mu_1 - \mu_2)^2) \equiv \phi_1(\theta_2).$$

where  $q=1-p$ , and

$$\hat{\theta}_2 = (\bar{Y}_1, \bar{Y}_2, S_{12}^2/N) \rightarrow \theta_2 = (\mu_1, \mu_2, \eta).$$

Now, we compute  $B_1^*(\theta_1)$  and  $B_2^*(\theta_2)$  in the intrinsic equations (2.2).

**Proposition 2.** The quantities  $B_1^*(\theta_1)$  and  $B_2^*(\theta_2)$  are

$$B_1^*(\theta_1) = \frac{\sqrt{pq}}{\pi} \cdot \frac{1}{\sqrt{\eta}} \text{ and } B_2^*(\theta_2) = \frac{\sqrt{pq}}{\pi} \cdot \frac{\sqrt{\eta}}{\eta + pq(\mu_1 - \mu_2)^2}. \quad (3.3)$$

**Proof.** The results immediately follow from the strong law of large numbers.

**Proposition 3.** As  $(\mu_1, \mu_2, \eta) \rightarrow (\mu, \mu, \eta)$ ,

$$B_2^*(\theta_2) \rightarrow B_1^*(\theta_1).$$

**Proof.** It is straightforward.

Now, from the results of Proposition 1 and 2 and the intrinsic equations given in (2.2), a set of the intrinsic priors is given by

$$\begin{cases} \pi_1^I(\mu, \eta) = g(\mu, \eta), \\ \pi_2^I(\mu_1, \mu_2, \eta) = \frac{\sqrt{pq}}{\pi\sqrt{\eta}} \times g(p\mu_1 + q\mu_2, \eta + pq(\mu_1 - \mu_2)^2), \end{cases} \quad (3.4)$$

where  $g(\mu, \eta)$  is any proper density for  $(\mu, \eta) \in \Theta_1$ .

**Theorem 1.** The intrinsic prior  $\pi_2^I(\mu_1, \mu_2, \eta)$  in (3.4) is proper on  $\Theta_2$ .

**Proof.** Let  $s = p\mu_1 + q\mu_2$  and  $t = \mu_1 - \mu_2$ . Then

$$\begin{aligned} & \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \pi_2^I(\mu_1, \mu_2, \eta) d\mu_1 d\mu_2 d\eta \\ &= \frac{\sqrt{pq}}{\pi} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{\eta}} \cdot g(s, \eta + pq t^2) ds dt d\eta. \end{aligned}$$

Let  $k_1 = \eta$  and  $k_2 = \sqrt{pq}t$ , and then letting  $u = \frac{k_2}{\sqrt{k_1}}$  and  $\nu = \sqrt{k_1}$ ,

$$\frac{2}{\pi} \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \nu \cdot g(s, \nu^2(1+u^2)) ds d\nu du.$$

Finally, letting  $w = \nu^2(1+u^2)$ , then

$$\frac{1}{\pi} \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty g(s, w) \frac{1}{1+u^2} ds dw du$$

and since  $g$  is a probability density function on  $\Theta_1$ , therefore

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{1+u^2} du = 1.$$

This completes the proof.

**Corollary 1.** When  $g(\mu, \eta)$  is given by  $g(\mu, \eta) = g_1(\mu | \eta)g_2(\eta)$  with  $\mu | \eta \sim N(\omega, x\eta)$  and  $\eta \sim IG(\alpha, \beta)$ , the set of intrinsic priors is given by

$$\begin{cases} \pi_1^I(\mu, \eta) = \frac{\beta^\alpha}{\sqrt{2\pi x} \Gamma(\alpha)} \left(\frac{1}{\eta}\right)^{\alpha + \frac{3}{2}} \exp\left[-\frac{1}{\eta} \left(\frac{(\mu - \omega)^2}{2x} + \beta\right)\right], \\ \pi_2^I(\mu_1, \mu_2, \eta) = \frac{\sqrt{pq}}{\pi\sqrt{\eta}} \times g(p\mu_1 + q\mu_2, \eta + pq(\mu_1 - \mu_2)^2), \end{cases} \quad (3.5)$$

where

$$\begin{aligned} g(p\mu_1 + q\mu_2, \eta + pq(\mu_1 - \mu_2)^2) &= \frac{\sqrt{pq}}{\pi\sqrt{\eta}} \frac{\beta^\alpha}{\sqrt{2\pi x} \Gamma(\alpha)} \left(\frac{1}{\eta + pq(\mu_1 - \mu_2)^2}\right)^{\alpha + 3/2} \\ &\times \exp\left[-\frac{1}{\eta + pq(\mu_1 - \mu_2)^2} \left(\frac{(p\mu_1 + q\mu_2 - \omega)^2}{2x} + \beta\right)\right]. \end{aligned}$$

**Corollary 2.** Using the intrinsic priors in Corollary 1, the Bayes factor of model  $M_2$  to model  $M_1$  is given by

$$B_{21}^I = \frac{m_2^I(\mathbf{x})}{m_1^I(\mathbf{x})}, \quad (3.6)$$

where

$$m_1^I(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{N/2} \frac{1}{\sqrt{1+xN}} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{T_1 T_2} \frac{\Gamma(\alpha + N/2)}{\left(\frac{N(\omega - \bar{y})^2}{2(1+xN)} + \beta + \frac{s^2}{2}\right)^{\alpha + N/2}},$$

and

$$\begin{aligned} m_2^I(\mathbf{x}) &= \left(\frac{1}{2\pi}\right)^{N/2} \frac{2}{\pi} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{T_1 T_2} \\ &\times \int_0^\infty \int_{-\infty}^\infty \left(\frac{1}{\nu}\right)^{N+2\alpha+1} \left(\frac{1}{1+u^2}\right)^{\alpha+1} \frac{1}{\sqrt{1+xN(1+u^2)}} \\ &\times \exp\left(-\frac{s_{12}^2}{2\nu^2} - \frac{\beta}{\nu^2(1+u^2)} - R(u, \nu)\right) du d\nu, \end{aligned}$$

respectively with

$$\begin{aligned} R(u, \nu) &= \frac{1}{2\nu^2(1+xN(1+u^2))} \left[ n_1(\bar{y}_1 - \sqrt{\frac{q}{p}} u\nu - \omega)^2 + n_2(\bar{y}_2 + \sqrt{\frac{p}{q}} u\nu - \omega)^2 \right. \\ &\left. + n_1 n_2 x(1+u^2) \left(\bar{y}_1 - \bar{y}_2 - \frac{u\nu}{\sqrt{pq}}\right)^2 \right]. \end{aligned}$$

**Proof.** It is straightforward.

### 4. Numerical results

In this section, we will give some examples to compare the fractional Bayes factors and the Bayes factors using the intrinsic priors via the simulation.

We performed the simulation study for testing  $M_1: \mu_1 = \mu_2 = 0$  and  $M_2: \mu_1 \neq \mu_2$  to examine how well Bayes factors using the intrinsic priors approximate to the fractional Bayes factors. The random numbers are generated from lognormal distribution with  $\mu_i = 0, i = 1, 2$  and common parameter  $\eta = 1$ . The simulation is replicated 100 times for some choices of  $(\alpha, \beta)$  and  $x$ , assuming that  $\omega = 0$ . We compute the relative errors  $|B_{21}^F - B_{21}^I| / B_{21}^F$  and standard deviations of relative errors for  $(\alpha, \beta) = (0.01, 0.01), (0.1, 0.1), (1.0, 1.0)$  and  $x = 5, 10, 50$ . The values of  $x$  are chosen by the intuitive reason that in Example 1, Bayes factors using the intrinsic priors for these values of  $x$  gives almost same values of the fractional Bayes factors. The results are reported in Table 1. The numbers in parentheses are the standard deviations of the relative errors.

Table 1. Absolute relative errors between  $B_{21}^F$  and  $B_{21}^I$

$(n_1, n_2)$	$x \backslash (\alpha, \beta)$	(0.01,0.01)	(0.1,0.1)	(1.0,1.0)
(10,10)	5	.10835(.01847)	.10791(.01841)	.10406(.02230)
(10,20)		.07541(.01751)	.07526(.01767)	.07387(.02010)
(20,20)		.06520(.02796)	.06534(.02799)	.06715(.02869)
(20,30)		.06091(.04268)	.06117(.04297)	.06481(.04486)
(30,30)		.08410(.05642)	.08453(.05677)	.08913(.06012)
(10,10)	10	.10861(.01840)	.10817(.01833)	.10431(.02223)
(10,20)		.07555(.01747)	.07539(.01763)	.07400(.02006)
(20,20)		.06527(.02796)	.06541(.02799)	.06721(.02871)
(20,30)		.06094(.04270)	.06120(.04298)	.06482(.04488)
(30,30)		.08411(.05643)	.08455(.05678)	.08913(.06016)
(10,10)	50	.10882(.01834)	.10838(.01827)	.10451(.02217)
(10,20)		.07566(.01744)	.07550(.01760)	.07410(.02004)
(20,20)		.06532(.02796)	.06547(.02799)	.06726(.02873)
(20,30)		.06097(.04271)	.06123(.04300)	.06484(.04490)
(30,30)		.08412(.05644)	.08456(.05679)	.08913(.06018)

We can see that relative errors are quite small for each simulated dataset. It is well known that the Bayes factors using noninformative priors are defined only up to arbitrary constants. The intrinsic Bayes factor and the fractional Bayes factor are proposed for removing this arbitrariness. The IBF approach have some advantages to be quite satisfactory in nonnested,



as well as nested, model comparison and any distribution. However, the IBF has difficulty to require a heavy computation because the number of training sample might be enormous for large sample size. But FBF's are easier to compute than IBF's and more effective in computational time-consuming. The FBF approach has disadvantage to be inadequate for very small sample size. So, the FBF is limited than IBF in application range. However, FBF is also well defined and seems to be reasonably close to actual Bayes factors. The numerical results show that the FBF's are asymptotically equivalent to the corresponding Bayes factors using the intrinsic priors.

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