Proceedings of Joint Conference of The Korean Data Analysis Society and Korean Data And Information Science Society, October 17-18, 2003, pp. 19-30

A General Class of Acceptance-Rejection Distributions and Its Applications*

Hea-Jung Kim, Joonkeun Yum, Yung-Seop Lee[1],
Chun-Ho Cho, Hyo-Sang Chung[2]

Abstract

In this paper we present a new family of distributions that allows a continuous variation not only from normality to non-normality but also from unimodality to bimodality. Its properties are especially useful in studying and making inferences about models involving the univariate truncated normal distribution. The properties of the family and its applications are given.

Key words and phrases : Family of distributions; moments of random function; functional binary regression; tolerance interval.

1. Introduction

A random variable X has a singly truncated normal distribution if its probability density function is

$$\sigma^{-1}\phi((x-\mu)/\sigma)[1-\Phi((A-\mu)/\sigma)]^{-1}, A \le x, \tag{1}$$

where A is the lower truncation point; $\phi(\cdot)$ and $\Phi(\cdot)$ denote the p.d.f. and d.f. of the $N(0,1)$ variable, respectively; the degree of truncation is $\Phi((A-\mu)/\sigma)$ from below. When $A=\mu$, (1) is actually the distribution of $\mu+\sigma|U|$ where U is the $N(0,1)$ variable. It is noted that $\mu+\sigma|U|$ variable is closely related to the so called skew-normal distribution: Azzalini (1985) and Henze (1986) worked on the distribution, a family of distributions including the standard normal, but with an extra parameter to regulate skewness. For independent standard normal variables, U and V, a random variable

$$Z=\frac{\theta}{\sqrt{1+\theta^2}}|U|+\frac{1}{\sqrt{1+\theta^2}}V \tag{2}$$

---

*This work was supported by KEOP project at METRI(Meteorological Research Institute)/KMA in 2003.
[1]Department of Statistics, Dongguk University, Seoul 100-715.
[2]Meteorological Research Institute, KMA, Seoul 156-720.

is said to be skew-normal with parameter θ, written $Z \sim SN(\theta)$, where the parameter θ which regulates the skewness varies in $(-\infty, \infty)$. Thus, $\theta = 0$ corresponds to the $N(0,1)$ variable and fixed V corresponds to the singly truncated normal. We refer Arnold et al. (1993), Azzalini and Valle (1996) and Chen, Dey and Shao (1999), Kim (2001) for the applications of the distribution.

Present paper discusses, in detail, some properties of the singly truncated normal distributions. The properties are mainly about expectations of some functions of the truncated normal variable X with distribution (1) (see, Nakamura 1980 and Sugjura and Gomi 1985, for the usual moments of X). They are obtained from utilizing the normalizing constant of a family of distributions which includes $SN(\theta)$ as a special case. This gives rich properties of the truncated normal distribution that allow various inference about the truncated normal population having the distribution (1). Moreover, such results are potentially relevant for practical applications, since in data analysis there are a few parametric models available to dealing with truncated data, especially for the problem of fitting truncated data. A particular application is that the results can be used to define a broad class of binary linear regression models on the provision that a link function in a generalized linear model can be either asymmetric or symmetric.

2. The Family of Distributions

This section derives a new family of distributions that is useful for studying the properties of expectation of $\Phi(X) = \Phi(\theta|U| + k)$, a function of X with density (1), and that of its variants.

Proposition 1. Let Z_1 and Z_2 be independent variables having respective p.d.f.'s $h_1(z_1)$ and $h_2(z_2)$, and let $T(y), y \in R$, be an arbitrary continuous function. Then the distribution of $Z \equiv Z_1 \mid Z_2 < T(Z_1)$, i.e. the conditional distribution of Z_1 given $Z_2 < T(Z_1)$, is

$$f_Z(z) = Ch_1(z) H_2(T(z)), \ -\infty < z < \infty, \qquad (3)$$

where $H_{Z_i}(\cdot)$ is the d.f. of Z_i and $C^{-1} = \int_{-\infty}^{\infty} h_1(z) H_2(T(z_1)) dz_1$.

Proof. By Bayes' formula

$$\begin{aligned} F_Z(z) &= \Pr(Z_1 \le z \mid Z_2 \le T(Z_1)) \\ &= \int_{-\infty}^{z} dz_1 \int_{-\infty}^{T(z_1)} h_1(z_1) h_2(z_2) dz_2 / \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{T(z_1)} h_1(z_1) h_2(z_2) dz_2. \end{aligned}$$

Differentiating $F_Z(z)$ with respect to z, we obtain

$$f_Z(z)=h_1(z)\int_{-\infty}^{T(z)} h_2(z_2)dz_2\left[\int_{-\infty}^{\infty} h_1(z)\int_{-\infty}^{T(z_1)} h_2(z_2)dz_2dz_1\right]^{-1}.$$

Proposition 1 implies that the random variable with density (3) can be generated by the following acceptance-rejection technique. Independently sample Z_1 and Z_2 from $h_1(y)$ and $h_2(y)$. If $Z_2 < T(Z_1)$ then put $Z=Z_1$. Otherwise restart sampling a new pair of variables Z_1 and Z_2 until the inequality satisfied.

Using Proposition 1, we can define various distributions. Some of them are as follows:

Note 1. For $T(Z_1)=\theta Z_1$, $-\infty<\theta<\infty$, if Z_1 and Z_2 are i.i.d. random variables symmetric about zero, then the normalizing constant of (3) is $C=2$, because $pr(Z_2<\theta Z_1)=1/2$ for $-\infty<\theta<\infty$.

For example, when Z_1 and Z_2 are independent $N(0,1)$ variables and $T(Z_1)$ is equal to θZ_1, $-\infty<\theta<\infty$, it is straightforward to see, from Henze (1986) that, the distribution of Z is $SN(\theta)$, and its density function is

$$f_1(z;\theta)=2\phi(z)\Phi(\theta z),\ -\infty<z<\infty.$$

For another example, if Z_1 and Z_2 are independent $t_{(\nu)}$ variables and $T(Z_1)=\theta Z_1$. Then, for $-\infty<\theta<\infty$, the distribution of $Z=Z_1|Z_2<\theta Z_1$ is

$$f_2(z;\theta)=2f_{(\nu)}(z)F_{(\nu)}(\theta z),\ -\infty<z<\infty,$$

where $f_{(\nu)}(\cdot)$ and $F_{(\nu)}(\cdot)$ are p.d.f. and d.f. of $t_{(\nu)}$ variable, respectively.

Note 2. If Z_1 and Z_2 are independent variables with respective p.d.f.'s $h_1(z_1)$ and $h_2(z_2)$. Then the normalizing constant $C^{-1}=E_{Z_1}[H_2(T(Z_1))]$.

For example, if $Z_1\sim N(0,1)$ and $Z_2\sim U(0,1)$ are independent random variables and $T(Z_1)=\theta Z_1^2$, then

$$f_3(z;\theta)=z^2\phi(z),\ -\infty<z<\infty,$$

independent of θ, because $H_2(\theta Z_1^2)=\theta Z_1^2$ for $-\infty<\theta<\infty$, and hence the normalizing constant is $C^{-1}=E_{Z_1}[\theta Z_1^2]=\theta$.

For another example, if $Z_1 \sim N(0,1)$ and $Z_2 \sim U(0,1)$ are independent random variables and $T(Z_1) = \theta\ |Z_1|$, then

$$f_4(z;\theta) = \frac{\sqrt{\pi}}{\sqrt{2}} |z\ |\phi(z), \ -\infty < z < \infty,$$

independent of θ, because $H_2(\theta|Z_1|) = \theta|Z_1|$ for $-\infty < \theta < \infty$, and hence the normalizing constant is $C^{-1} = E_{Z_1}[\theta\ |Z_1|] = \sqrt{2}\theta/\sqrt{\pi}$.

Shapes of example densities are given in Figure 1.

Note 3. When Z_1 and Z_2 are independent t variables with respective degrees of freedom ν_1 and ν_2, then, for $T(Z_1) = \theta Z_1$ the distribution of Z is the skew-t distributions introduced by Branco and Dey (2001) and Kim (2002); The skewed-Pearson type II distribution by Branco and Dey (2001) also has the density of form (3), if we assume that Z_1 and Z_2 are independent Pearson type II variables with appropriate parameters.

The family $\{f_T; T(y), y \in R\}$ defined by (3) denotes a family of conditional densities. As seen in the above examples and Figure 1 below, the interest in the family comes from two directions. On theoretical side, it enjoys a number of formal properties which reproduce or resemble both symmetric and skewed distributions. From applied viewpoint, the family leads to

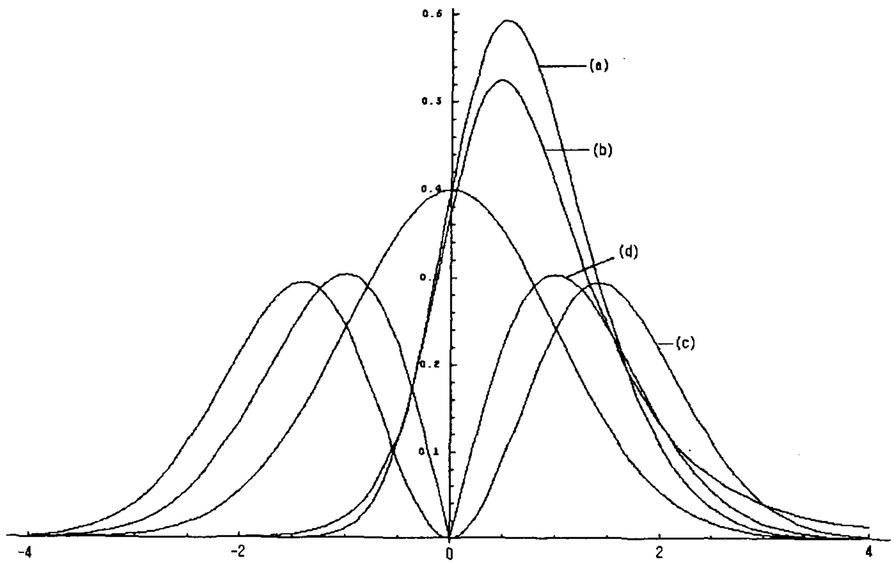

Figure 1. Shapes of example densities: (a) $f_1(z;\theta=2)$; (b) $f_2(z;\theta=2, \nu=5)$; (c) $f_3(z;\theta=2)$; (d) $f_4(z;\theta=2)$.

skewed and possibly heavy-tailed distributions so that it is suitable for analysis of data exhibiting a unimodal empirical distribution but with some skewness present, a situation often occurring in practical problems. Furthermore, the family also leads to bimodal distributions that can be used for analysis of data having a bimodal empirical distribution.

3. Moments of Functions of Standard Normal Variable

When Z_1 and Z_2 are independent $N(0,1)$ variables, the family of distributions $\{f_T; T(y), y \in R\}$ defined by (3) is useful for studying yet another properties of random functions of $N(0,1)$ variable. The properties are as follows.

Proposition 2. If U is a $N(0,1)$ random variable, then

$$E[\Phi(\theta\,|U|+k)] = \Phi(k/\sqrt{1+\theta^2}) + 2G(k/\sqrt{1+\theta^2}, \theta) \tag{4}$$

for any real θ and k, where $G(a,b)$ is the function studied by Owen (1956) which gives the integral of the standard normal bivariate density over region bounded by lines $x=a$, $y=0$, and $y=bx$ in the (x,y) plane.

Proof. Let $V \sim N(0,1)$ independently of U. Then

$$\begin{aligned} E[\Phi(\theta|U|+k)] &= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\theta x+k} e^{-v^2/2}dv\right]e^{-x^2/2}dx \\ &= E_X[pr(V \le \theta x + k|x)] \\ &= pr_{V,|U|}(V-\theta\,|U| \le k) \\ &= pr_{V,|U|}((V-\theta\,|U|)/\sqrt{1+\theta^2} \le k/\sqrt{1+\theta^2}). \end{aligned}$$

For computing the last probability, we use of the distribution defined by (2) so that $Z=(V-\theta\,|U|)/\sqrt{1+\theta^2} \sim SN(-\theta)$, a skew-normal random variable with parameter- θ. Using the distribution function of W by Azzalini (1985), we see that the last probability is equivalent to $pr(Z \le k/\sqrt{1+\theta^2}) = \Phi(k/\sqrt{1+\theta^2}) + 2G(k/\sqrt{1+\theta^2}, \theta)$.

A computer routine which evaluates $G(a,b)$ has been given by Young and Minder (1974).

Corollary 1. If U is a $N(0,1)$ random variable, then

$$E[\Phi(\theta U + k)] = \Phi(k/\sqrt{1+\theta^2}). \tag{5}$$

Proof. The proof is immediate if we change $|U|$ with U in the above proof.

Proposition 2 and Corollary 1 give following properties.

Property 1. For $U \sim N(0,1)$ and $-\infty < \theta < \infty$, $E[\Phi(\theta|U|)] = pr(t_{(1)} \le \theta) = 1/2 + (\tan^{-1}\theta)/\pi$, because $G(0,\theta) = (\tan^{-1}\theta)/2\pi$. Thus

$$E[\theta\{\Phi(|U|) - \Phi(U)\}] = (\tan^{-1}\theta)/\pi.$$

Note that $E[\Phi(|U|)] = 3/4$. This gives $E[\Phi(|U|) - \Phi(U)] = 1/4$.

Property 2. For $U \sim N(0,1)$, the moment generating function of $X = \sigma|U| + \mu$ is

$$M_X(t) = 2e^{\mu t + \sigma^2 t^2/2}\Phi(\sigma t) \text{ for } -\infty < t < \infty.$$

Hence, after some algebra, we obtain $E[X] = \mu + \sigma(2/\pi)^{1/2}$ and $V(X) = \sigma^2(\pi - 2)/\pi$.

Proposition 3. If U is a $N(0,1)$ random variable, then for $\theta \ge 0$,

$$E[\Phi(|\theta U + k|)] = 1 - 2G(k/\sqrt{1+\theta^2}, 1/\theta) \tag{6}$$

Proof. Let $k = \theta\xi$, then for $\theta \ge 0$,

$$E[\Phi(|\theta U + k|)] = \int_{-\xi}^{\infty}\phi(y)\Phi(\theta(y+\xi))dy + \int_{\xi}^{\infty}\phi(y)\Phi(\theta(y-\xi))dy. \tag{7}$$

Let $L(\alpha, \beta, \rho) = pr(X_1 > \alpha, X_2 > \beta)$ for bi-variate standard normal variables X_1 and X_2 with correlation coefficient ρ. Putting $\theta y + \theta\xi = (\rho y - \alpha)/\sqrt{1-\rho^2}$, we see that the right hand side of the equation (7) is equal to

$$L(-\frac{\theta y}{\sqrt{1+\theta^2}}, -t, \frac{\theta}{\sqrt{1+\theta^2}}) + L(\frac{\theta y}{\sqrt{1+\theta^2}}, t, \frac{\theta}{\sqrt{1+\theta^2}}).$$

Using the relation between $L(\cdot)$ and $\Phi(\cdot)$ and $G(\cdot)$ functions, we have at once the result (see Sowden and Ashford 1967).

Proposition 3 and Corollary 1 give following properties.

Property 3. For $U \sim N(0,1)$, and $-\infty < k < \infty$,

$$E[\Phi(|U+k|)] = 1 - 2G(k/\sqrt{2}, 1) = 1 - \Phi(k/\sqrt{2})\Phi(-k/\sqrt{2}).$$

Property 4. For $U \sim N(0,1)$,

$$pr(|U| \le z) = 2\Phi(z) - 1.$$

Property 5. For $Y \sim N(\mu, \sigma^2)$,

$$E[\Phi(|Y|) - \Phi(Y)] = 1 - \{2G(\mu/\sqrt{1+\sigma^2}, 1/\sigma) + \Phi(\mu/\sqrt{1+\sigma^2})\}.$$

4. Applications

As seen in section 2, the family of distributions $\{f_T; T(y), y \in R\}$ defined by (3) includes various new unimodal and bimodal distributions yet to be studied. In this section the distribution are applied to a generalized regression model. Moreover, some result in section 3 are applied to certain statistical techniques.

4.1. Application to a Functional Binary Regression Model

In this section, we consider a functional modelling of a generalized regression model. We restrict our attention to, for simplicity, a single explanatory variable, although similar ideas can be applied to multiple explanatory variables. Let $Y = (y_1, \cdots, y_n)'$ denote an $n \times 1$ vector of n independent dichotomous random variables. Also, let $\mathbf{x}_i(1, x_i)'$, be a 2×1 vector of covariates, and $\beta = (\beta_1, \beta_2)'$ is a 2×1 vector of regression coefficients. Assume that $y_i = 1$ with probability p_i and $y_i = 0$ with probability $1 - p_i$. In a traditional dichotomous quantal response model, it is usually assumed that

$$p_i = F(\mathbf{x}'_i \beta), \tag{8}$$

where $F(\cdot)$ denotes a c.d.f. and F^{-1} is typically called a link function in a generalized linear model setup. When F is a c.d.f. of symmetric distribution, the resulting link is symmetric. The most common symmetric links include the probit, logit, and t links which are the members of $\{f_T; T(y), y \in R\}$. Of course, as seen in figure 1, an asymmetric link can be obtained by taking an asymmetric distribution in $\{f_T; T(y), y \in R\}$. The complementary log-log link is an example of an asymmetric link. See Stukel (1988), Chen, Dey and Shao (1999), and Kim (2002) for the other classes of asymmetric links.

In this section we propose an alternative link model for dichotomous quantal response model. Specifically, our model considers a binary regression model with error in the explanatory variable. The model is motivated by a functional regression model studied by Kim

(2002) and Carroll et al. (1995), where the explanatory variable observations are considered to be realizations from a random variable.

Let $W=(w_1,\cdots,w_n)'$ be a vector of latent variables. Then, utilizing a latent variable approach of Albert and Chip (1993), our proposed functional binary regression model is formulated as

$$y_i=\begin{cases} 0 \text{ if } w_i<0 \\ 1 \text{ if } w_i\geq 0, \end{cases} \tag{9}$$

where

$$w_i=\beta_1+x_i\beta_2+\varepsilon_i, \qquad \varepsilon_i\sim H_1, \tag{10}$$

and

$$X_i=x_i+\theta e_i, \qquad e_i\sim H_2,$$

where ε_i and e_i are independent with c.d.f.s H_1 and H_2, respectively, and the explanatory variables are subject to independent measurement error, with only X_i being observed.

An immediate example of the error structure in the explanatory variable is radiance measurements from satellite-borne infrared sensors. It is apparent that measurement errors occur in true radiance readings, because the measurements are severely distorted by the presence of clouds in the fields of view of the sensors (see, DePrist 1983 and Azzalini and valle 1996 for other examples).

As was assumed if errors in explanatory variables are known to present in terms of the functional form, then the most appropriate modelling and fitting procedure will take account of the full error structure. In what follows we reparameterize the model (10) in terms of β_1, β_2, $\delta=-\beta_2\theta$. That is

$$w_i=\beta_1+X_i\beta_2+\delta e_i+\varepsilon_i, \qquad \varepsilon_i\sim H_1, \quad e_i\sim H_2. \tag{11}$$

If H_1 is the c.d.f. of a symmetric distribution, (10) and (11) give the functional binary response model

$$p_i=pr(y_i=1)=\int_{-\infty}^{\infty}H_1(\beta_1+\beta_2X_i+\delta e_i)h_2(e_i)de_i=E_{e_i}[H_1(\beta_1+\beta_2X_i+\delta e_i)]. \tag{12}$$

Let $D_{obs}=\{y_i,X_i:\ i=1,\cdots,n\}$ denote the observed data. Then, from (11), the likelihood function for the model is given by

$$L(\beta_1,\beta_2,\delta|D_{obs})=\prod_{i=1}^{n}\int_{-\infty}^{\infty}[H_1(\beta_1+\beta_2X_i+\delta e_i)]^{y_i}[1-H_1(\beta_1+\beta_2X_i+\delta e_i)]^{1-y_i}h_2(e_i)de_i.$$

(9) and (11) defines a rich class of symmetric and asymmetric link models. For example, suppose ε_i and e_i in (11) are independent $N(0,1)$ variables in (10), we see that Corollary 1 leads to

$$p_i = pr(y_i = 1) = \Phi((\beta_1 + X_i\beta_2)/\sqrt{1+\delta^2}),$$

a usual symmetric probit model with scale change in the regression coefficients.

For another example, when $\varepsilon_i \sim N(0,1)$ and $e_i = |U_i|$ in (10) are independent variables , where $U_i \sim N(0,1)$. Then Proposition 2 gives

$$p_i = E[\delta|U| + \beta_1 + X_i\beta_2] = \Phi((\beta_1 + X_i\beta_2)/\sqrt{1+\delta^2}) + 2G((\beta_1 + X_i\beta_2)/\sqrt{1+\delta^2}, \delta),$$

a asymmetric link model, so called skew-normal link model which was derived by Chen, Dey, and Chao (1999).

Last example is the skew- t link model by Kim (2002). It is derived when the c.d.f.'s $(H_1, H_2) \in \Omega$, where

$$\Omega = \{(H_1, H_2) : \varepsilon_i \sim N(0, \lambda^{-1}),\ e_i \sim N(0, \lambda),\ \lambda \sim \Gamma(\nu/2, 2/\nu) \text{ with } E[\lambda] = 1\}.$$

Analysis of those example models are well developed in the referred references, and hence it is omitted here.

4.2. Application to β-Expectation Tolerance Intervals

A β-expectation tolerance interval for the normal distribution is a random interval which includes, on the average, a proportion β of the normal population (see Zacks 1971, pp. 516). The interval may be of the form $(-\infty, I_U]$, or $[I_L, \infty)$ or $[I_U, I_L]$. In the case of an interval of the form $(-\infty, I_U]$ for a $N(\mu, \sigma^2)$ distribution, for example, a statistic I_U is required such that $E[\Phi((I_U - \mu)/\sigma)] = \beta$ in the unknown parameter(s). Corollary 1 and Proposition 3 are used below to derive the statistic I_U in $(-\infty, I_U]$ for the case of unknown μ and σ known. β-expectation tolerance intervals of form $[I_L, \infty)$, or $[I_U, I_L]$ are easily obtained from the results given for the intervals of the form $(-\infty, I_U]$.

Example 1. β-Expectation Tolerance Interval of μ under the distribution of $\overline{X}$: Given an observation $\overline{x}$ on $\overline{X} \sim N(\mu, \sigma^2/n)$, a statistic $I_U(\overline{x}, \sigma)$ is required such that

$$E\left[\Phi\left(\frac{U(\overline{X},\sigma)-\mu}{\sigma}\right)\right]=\beta.$$

If $I_U(\overline{x},\sigma)=a\overline{x}+b$, then it is required that, by Corollary 1,

$$\beta=\Phi(K_\beta)=E\left[\Phi\left(\frac{a\overline{X}+b-\mu}{\sigma/\sqrt{n}}\right)\right]=\Phi\left(\frac{\sqrt{n}\{(a-1)\mu+b\}}{\sigma\sqrt{1+a^2}}\right).$$

Equating of coefficients in

$$K_\beta=\frac{\sqrt{n}\{(a-1)\mu+b\}}{\sigma\sqrt{1+a^2}}$$

immediately yields $a=1$ and $b=\sigma\sqrt{2/n}K_\beta$.

Example 2. β-Expectation Tolerance Interval of μ under the distribution of $|\overline{X}|$: Given an observation $\overline{x}$ on $\overline{X}\sim N(\mu,\sigma^2/n)$, a statistic $I_U(\overline{x},\sigma)$ is required such that, under the distribution $|\overline{X}|$,

$$E\left[pr\left(\frac{|\overline{X}-\mu|}{\sigma/\sqrt{n}}\le\frac{|I_U(\overline{X},\sigma)-\mu|}{\sigma/\sqrt{n}}\right)\right]=\beta.$$

If $I_U(\overline{x},\sigma)=a\overline{x}+b$, then it is required that

$$\beta=E\left[2\Phi\left(\frac{|a\overline{X}+b-\mu|}{\sigma/\sqrt{n}}\right)-1\right]=1-4G\left(\frac{\sqrt{n}\{(a-1)\mu+b\}}{\sigma\sqrt{1+a^2}},\,1/a\right)$$

by Properties 3 and property 4. Putting $a=1$, we have, by prooerty 3,

$$\beta=1-2\Phi(K_\beta^*)\Phi(-K_\beta^*), \tag{3}$$

where $K_\beta^*=b/\{\sigma\sqrt{2/n}\}$ and the value of K_β^* can easily obtained by solving(13).

Equating of coefficients in K_β^* immediately yields $a=1$ and $b=\sigma\sqrt{2/n}K_\beta^*$.

5. Concluding Remark

This paper has proposed a new family of distributions, denoted by $\{f_T;T(y),y\in R\}$. It is a parametric class of probability distributions came from $Z=Z_1\mid Z_2<T(Z_1)$, where Z_1 and Z_2 are independent continuous random variables and have respective p.d.f.s $h_1(z_1)$ and $h_2(z_2)$. The special feature of the family is that it gives a rich family of parametric density

functions that allow a continuous variation from normality to non-normality. Therefore the family of distributions is potentially relevant for practical applications, especially for the analysis of skewed data and bimodal data. Immediate applications of the distribution can be illustrated as follows: (i) Binary regression with an asymmetric link function; (ii) tolerance interval estimation under normal or truncated normal distribution; (iii) regression analysis with measurement errors in explanatory variables. A study pertaining to the applications is an interesting research topic and it is left as a future study of interest.

References

[1] Albert, J. and Chib, S. (1993). Bayesian analysis of binary and polytomous response data, *Journal of the American Statistical Association*, 88, 669-679.

[2] Arnold, B. C., Beaver, R. J., Groeneveld, R. A. and Meeker, W. Q. (1993). The nontruncated marginal of a truncated bivariate normal distribution, *Psychometrica*, 58, 471-478.

[3] Azzalini, A. (1985). A class of distributions which includes the normal ones, *Scandinavian Journal of Statistics*, 12, 171-178.

[4] Azzalini, A. and Valle, A. D. (1996). The multivariate skew-normal distribution, *Biometrika*, 83, 715-726.

[5] Branco. M. D. and Dey, D, K. (2001). A general class of multivariate skew-elliptical distribution, *Journal of Multivariate Analysis*, 79, 99-113.

[6] Carroll, R. J., Ruppert, D. and Stefanski, L. A. (1995). *Measurement Error in Nonlinear Models*, Chapman and Hall, London.

[7] Chen, M. H., Dey, D. K., and Shao, Q. M. (1999). A new skewed link model for dichotomous quantal response model, *Journal of the American Statistical Association*, 94, 1172-1186.

[8] Depriest, D. J. (1983). Using the singly truncated normal distribution to analyze satellite data, *Communications in Statistics - Theory and Methods, 12*, 263-272.

[9] Henze, N. (1986). A probabilistic representation of the 'Skewed-normal' distribution, *Scandinavian Journal of Statistics*, 13, 271-275.

[10] Kim, H. J. (2001). Bayesian analysis of a new skewed multivariate probit model for correlated binary response data, *Journal of the Korean Statistical Society*, 30, 613-635.

[11] Kim, H. J. (2002). Binary regression with an class of skewed t link models, *Communications in Statistics - Theory and Methods*, 31, 1863-1886.

[12] Nakamura, T. (1980). On the moment of positively truncated normal distribution, *Journal of the Japan Statistical Society*, 10, 139-144.

[13] Owen, D. B. (1956). Tables for computing bi-variate normal probabilities, *Annals of Mathematical Statistics*, 27. 1075-1090.

[14] Sowden, R. R. and Ashford, J. R. (1967). Computation of the bi-variate normal integral, *Journal of the Royal Statistical Society*, 30, 613-635.

[15] Sugjura, N. and Gomni, A. (1985). Pearson diagrams for truncated normal and truncated Weibull distributions, *Biometrika*, 72, 219-222.

[16] Zacks, S. (1981). *Parametric Statistical Inference*, Pregamon Press, Oxford.