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Optimal Designs for Attribute Control Charts

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Abstract

Shewhart-type control charts have historically been used for attribute data, though they have ARL biased property and even are unable to detect the improvement of a process with some process parameters. So far most efforts have been made to improve the performance of attribute control charts in terms of faster detection of special causes without increasing the rates of false alarm. In this paper, control limits are proposed that yield an ARL (nearly) unbiased chart for attributes. Optimal design is also proposed for attribute control charts under a natural sense of criterion.

Key words : Shewhart-type chart, Attribute chart, Average Run Length, ARL-unbiased

1. Introduction

Attribute control charts have historically been used with 3-sigma limits for Shewhart charts. Woodall (1997) has pointed out that they may not result in an acceptable chart performance since most of the statistics used in attribute charts have right-skewed distributions. For skewed distributions, the ARL will increase initially as one moves from the in-control parameter value in the direction of the skewness. On account of this reason, most of the attribute charts are not ARL unbiased, of which the concept was first introduced by Pignatiello (1995). He suggested in his paper that a control chart is said to be ARL unbiased if the ARL curve attains its maximum at in-control value. What is more, Shewhart-type charts perform the poorest in the tails of the distribution when the process parameters are small. Ryan and Schwertman (1997) suggested Total Absolute Difference Criterion (TADC) for optimal designs for attributes. Optimal design is obtained by minimizing

$$\left|\frac{1}{LTA} - \frac{1}{0.00135}\right| + \left|\frac{1}{UTA} - \frac{1}{0.00135}\right|$$

, where $LTA = P(X < LCL)$ and $UTA = P(X > UCL)$. Note that 0.00135 is the probability that when the data is exactly distributed from Standard normal distribution, the signal will occur beyond the upper control limit (or below the lower control limit). On the

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other hand, many researchers such as Simon (1941), Ryan (1989), and Acosta-Mejia (1999) and have suggested the alternatives to overcome the difficulty that a Shewhart chart for attributes does not always have a lower control limit.

All continuous designs, including the Shewhart-type design and ARL-unbiased design, give the charts the same false alarms rate, that is, in-control ARL due to the constraint,

$$1 - P(LCL \le X \le UCL \mid \theta = \theta_0) = \alpha \qquad (1)$$

Thus continuous control charts have the different ARLs at the parameter values except for in-control value. It means that continuous charts which have the equal operation at in-control have the various performances at out-of-control values. Unfortunately this property can not come under the attribute charts. In fact it is impossible for all the attribute charts to satisfy (1) for any fixed α. For this reason, a fair comparison with performances of attribute charts may not be made. Note that the false alarms rate is a kind of type I and it must be controlled under any circumstances of quality process.

In this paper, new design for attribute charts is suggested, including the constraint of controlling the false alarms rate. Under these designs for attribute charts, ARL unbiased design and optimal design for attribute charts are proposed on the basis of their theoretical studies and their performances are compared with other attribute charts through the ARL values.

2. Design for Attribute Charts

Let X be the random variable distributed from process parameter θ. Through monitoring the value of θ, the state of the present process is easily catched out as in-control or out-of-control. Assume that the in-control value of the process is θ_0. Then if $\theta = \theta_0$, the process is in control, otherwise the process has shifted.

$[L, U : \alpha^*]$ is said to be a design for attribute chart if

(D1) $0 \le L < U < \infty$, L, U : integer ; and

(D2) for any fixed L, U is the minimum of values that satisfy

$$1 - P(L \le X \le U \mid \theta = \theta_0) \le \alpha^* \qquad (2)$$

, in which α^* is a prefixed permissible maximum limit of the false alarms rates. In this paper, α^* is fixed as the rate of false alarms for the 3-sigma limits for Shewhart chart, that is, $\alpha^* = 1 - P(L^S \le X \le U^S)$, where L^S and U^S are the lower and upper control limit respectively for 3-sigma chart. Due to the fact that Type I error and Type II error have inverse proportion relation, condition (D2) confirms that for any fixed lower control limit L, the design have the most powerful performance of the attribute charts of which the rates of false alarms are less than or equal to upper bound α^*.

As the false alarms rate is controlled by its upper bound, the design has bounds on lower and upper control limits. In order to search the upper and lower bounds of them, the following definitions on limiting designs are needed.

Assume that $[L^*, U_{L^*} : \alpha^*]$ be the design that satisfies;

(i) L^* is maximum of non-negative integers which satisfy $P(X < l \mid \theta = \theta_0) \le \alpha^*$; and

(ii) for such L_*, U_{L^*} is minimum of positive integers which satisfy

$$1 - P(L^* \le X \le u \mid \theta = \theta_0) \le \alpha^*.$$

Similarly $[L_{U^*}, U^* : \alpha^*]$ can be assumed to be the design that satisfies two conditions:

(i) U^* is minimum of positive integers which satisfy $P(X > u \mid \theta = \theta_0) \le \alpha^*$; and

(ii) for such U^*, L_{U^*} is maximum of non-negative integers which satisfy

$$1 - P(L^* \le X \le u \mid \theta = \theta_0) \le \alpha^*.$$

Now we can easily show that the following relation holds.

$$0 \le L_{U^*} \le L \le L^* < U^* \le U \le U_{L^*}.$$

3. ARL (Nearly) Unbiased Design for np Control Charts

Let X be the number of units of product that are nonconforming, having a binomial distribution with parameter n and p. Assume that the in-control value of the process is p_0. For a given design, certain properties of the ARL curve can be identified.

Let $[L, U: \alpha^*]$ be a design for np chart and set

$$p^* = \left[\left(\frac{(U+1)\binom{n}{U+1}}{L\binom{n}{L}} \right)^{\frac{1}{U-L+1}} + 1 \right]^{-1} \tag{3}$$

Then $ARL(p)$ is monotonically increasing in $0 < p < p^*$ and monotonically decreasing in $p^* < p < 1$. Thus $ARL(p)$ attains its maximum at p^*. p^U is the value of p^* in case of $[L^*, U_{L^*}: \alpha^*]$ and p^L is the value of p^* in case of $[L_{U^*}, U^*: \alpha^*]$. Then p^* increases as L increases (that is, $L \to L^*$), and p^* converges to p^L (p^U) as $L \to L_{U^*}$ ($U \to U_{L^*}$). That is, p^* is a monotonous increasing function of L and it has the value between p^L and p^U (that is, $p^L < p^* < p^U$). A pair of limits that conforms with Equation (3) at $p^* = p_0$ produces an np chart with ARL unbiased performance. However it is not possible to comply with Equation (3) for all p_0. For such cases it will be convenient to use control limits that produce nearly unbiased performance. The objective is to look for a pair of limits which produce the least difference between the right and left side of Equation (3). For any integers L and U with $0 \le L < U \le n$, $u(L, U; p_0)$ is defined as difference between the in-control value p_0 and the parameter value at which ARL unbiased property can be taken;

$$u(L, U; p_0) = \left[\left(\frac{(U+1)\binom{n}{U+1}}{L\binom{n}{L}}\right)^{\frac{1}{U-L+1}} + 1\right]^{-1} - p_0$$

$[L^U, U^U: \alpha^*]$ is said to be ARL (nearly) unbiased design if it minimizes

$$|u(L, U; p_0)| = \left|\left[\left(\frac{(U+1)\binom{n}{U+1}}{L\binom{n}{L}}\right)^{\frac{1}{U-L+1}} + 1\right]^{-1} - p_0\right| \quad (4)$$

for any design $[L, U: \alpha^*]$ for np chart.

4. ARL (Nearly) Unbiased Chart for c Control Charts

Let X is the number of nonconformities of a specified type, assumed to have a Poisson distribution with mean λ. The in-control value of the process is λ_0. Let $[L, U: \alpha^*]$ be a design for c chart and set

$$\lambda^* = \left(\frac{\Gamma(U+1)}{\Gamma(L)}\right)^{\frac{1}{U-L+1}} \quad (5)$$

Then $ARL(\lambda)$ is monotonically increasing, if $\lambda < \lambda^*$and monotonically decreasing, if $\lambda > \lambda^*$. Thus $ARL(\lambda)$ attains its maximum at λ^*. λ^U is the value of λ^* in case of $[L^*, U_{L^*}: \alpha^*]$ and λ^L is the value of λ^* in case of $[L_{U^*}, U^*: \alpha^*]$. Then λ^* increases as L increases and λ^* converges to λ^L (λ^U) as $L \to L_{U^*}$ $(U \to U_{L^*})$. Thus λ^* is a monotonous increasing function of L and it has the value between λ^L and λ^U (that is, $\lambda^L < \lambda^* < \lambda^U$). For the same reason as the case of the np chart, the method is suggested for deriving ARL nearly unbiased limits. For $0 \le L < U \le n$: integer, $u(L, U; \lambda_0)$ is defined as difference between the in-control value λ_0 and the parameter value at which ARL unbiased property can be taken;

$$u(L, U; \lambda_0) = \left(\frac{\Gamma(U+1)}{\Gamma(L)}\right)^{\frac{1}{U-L+1}} - \lambda_0.$$

$[L^U, U^U: \alpha^*]$ is said to be ARL (nearly) unbiased design if it minimizes

$$|u(L, U; \lambda_0)| = \left|\left(\frac{\Gamma(U+1)}{\Gamma(L)}\right)^{\frac{1}{U-L+1}} - \lambda_0\right| \quad (6)$$

for any design $[L, U: \alpha^*]$ for c chart.

5. Optimal Design for Attribute Charts

Zhang (2003) proposed optimal criterion for S^2 control chart and called it the Smallest

Area Criterion (SAC). We will find out the basic ideas and weak points of the SAC through the expansion of it to Attribute charts. Now define the limiting ARL curve (that is, the lower bound of all ARL curves). When an ARL is equal to the limiting curve at some process parameter values, it means that the performance of the control chart is optimized at that values. Let the limiting ARL be defined as

$$ARL^*(\theta)=\begin{cases}(1-P(L^*\le X\le U_{L^*}))^{-1}, & \text{for } 0\le\theta<\theta_0\\ 1/\alpha^* , & \text{for } \theta=\theta_0\\ (1-P(L_{U^*}\le X\le U^*))^{-1}, & \text{for } \theta>\theta_0\end{cases}$$

$ARL^*(\theta)$ is discontinuous at $\theta=\theta_0$, but this fact makes no matter of consequence because ARL values at the out-of-control are the subject of investigation. It can be easily shown that for any design $[L, U:\alpha^*]$, $ARL(\theta;L,U)\ge ARL^*(\theta)$, for $\theta\ge 0$ and $\theta\ne\theta_0$.

The first optimal criterion (***Criterion I***) is to derive the optimal design $[L^{O_1}, U^{O_1}:\alpha^*]$ by minimizing

$$\int_0^\infty (ARL(\theta;L,U) - ARL^*(\theta))d\theta \tag{7}$$

for any design $[L, U:\alpha^*]$. The idea is to use the area between the ARL curve for a possible design and the limiting ARL curve defined above to measure the out-of-control ARL performance. The smaller this area, the better the design performance. This idea behind the criterion is a natural one. ***Criterion I*** suggests as ideal design the chart that has the least amount of its ARL curve away from ARL curve for limiting limits. Thus the optimal design must be affected by RL distribution, and so by the original data distribution. As the results, if ***Criterion I*** is applied to attribute data, which usually have right-skewed distributions, the ARL curve has the peak at the larger parameter value than the in-control value. This phenomenon is the opposite to that of a Shewhart c chart, which has the ARL curve with the maximum at the left of in-control value. Due to these reasons, design by ***Criterion I*** has two weak points. One is that it is ARL biased. The other is that it has the poor performance in the neighborhood of the in-control value. Therefore the update design is required to be as close as possible to ideal ARL unbiased chart and perform the fast insight into any shift near the in-control value within the limits of the possible.

The gap between the ARL curve of the specific chart and that of the limiting chart can be considered as the room of performance which will be improved by the better criterion. It is worth noticing that the ARL performances of the charts have significant differences within the some radius away from the in-control value, but as the process shifts largely away from both sides of the in-control value, they converge to nearly the same value. Therefore new criterion is focused on the significant gap (in particular, the maximum gap), not on the total amount of the difference. Now we can define the another optimal criterion (***Criterion II***) based on the minimax concept. The optimal design $[L^{O_2}, U^{O_2}:\alpha^*]$ is derived by minimizing

$$\max_{\theta\ge 0}\ [ARL(\theta;L,U) - ARL^*(\theta)] \tag{8}$$

for any design $[L, U:\alpha^*]$ for attribute chart.

6. Comparison of Performances

Suppose it is desirable to monitor the number of non-conformities of an in-control process with λ_0. A rate of false alarms is limited by that of the 3-sigma limits for Shewhart chart. c charts which are considered here are Shewhart chart with 3-sigma limits, ARL(nearly) unbiased chart, designs by Criterion I and II.
Table 1 shows control limits for them if λ_0 is from 11.42 to 11.56. Figure 1 shows that ARL curves change dynamically though λ_0 increases bit by bit.
The bottom is the limiting ARL curve.
We can see two points in Figure 1.
The first point is that C2 curve is robust over in-control parameter, that is, the design by Criterion II has less effects on RL distribution than any other designs. The second is that Criterion II provides an ARL biased design of which the degree of ARL biasedness is as small as tolerated. Note that designs by Criterion II in figure 1 are all ARL (nearly) unbiased for c charts.

Table 1. Control limits for c charts

λ_0	Criterion for ***c*** control chart							
	(S) 3-sigma		(A) ARL unb.		(C1) Cri. I		(C2) Cri. II	
11.42	2	21	3	22	2	21	3	22
11.43	2	21	3	22	4	26	3	22
11.44	2	21	3	22	4	25	3	22
11.45	2	21	3	22	4	25	3	22
11.46	2	21	3	22	4	25	3	22
11.47	2	21	3	22	4	24	4	24
11.48	2	21	3	22	4	24	4	24
11.49	2	21	3	22	4	24	4	24
11.50	2	21	3	22	4	24	4	24
11.51	2	21	3	22	4	24	4	24
11.52	2	21	3	22	4	24	4	24
11.53	2	21	3	22	4	24	4	24
11.54	2	21	3	22	4	24	4	24
11.55	2	21	3	22	4	24	4	24
11.56	2	21	4	23	4	23	4	23

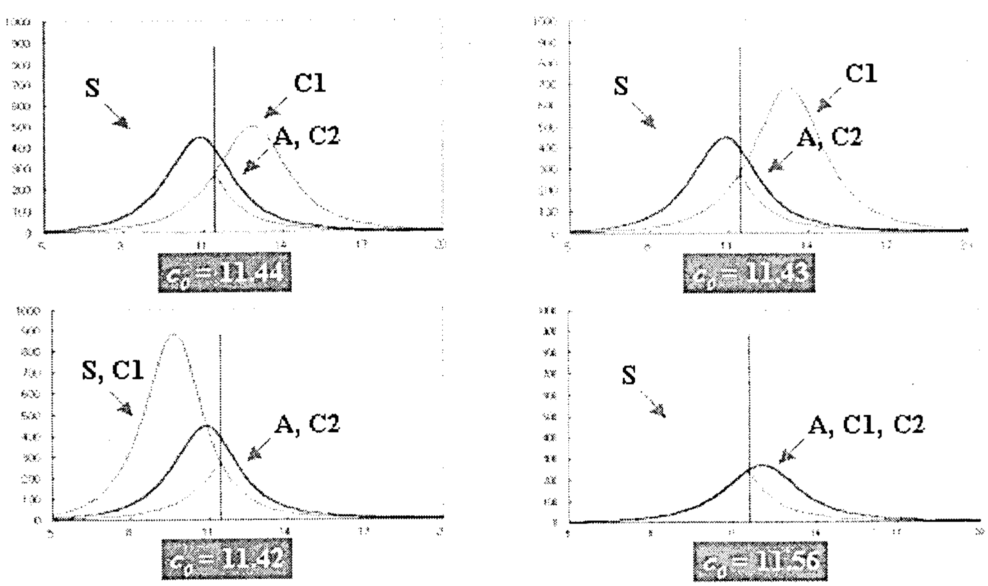

Figure 1. ARL curves for c charts

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