Test of Symmetry against Near Type III Positive Biasedness

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Abstract

One of the widely accepted assumptions in many statistical problem is that the underlying distribution is symmetric. Though a large number of nonparametric test are available in the literature for this problem, very few procedures focuses on the distributional structure when the symmetry assumption is rejected. Yanagimoto and Sibuya (1972) provided the various types of asymmetric distributional structure, positive biasedness, namely. In this paper we consider the test of symmetry against several new positive biasedness restrictions which are stronger than Yanagimoto and Sibuya's type II bias but weaker than type IV (III) bias.

Keywords: Isotonic regression, order restricted inference, peakedness, stochastic ordering.

1. Introduction

One of the widely accepted assumptions in many statistical problem is that the underlying distribution is symmetric. Many statistical procedures, for example Wilcoxon's signed rank test, may result in low validity if this symmetry assumption is violated. Moreover, when the symmetry assumption is satisfied many statistical procedures based on normal theory can be applied for many problems with moderate or large sample sizes. In this sense the symmetry assumption is very crucial. Though a large number of nonparametric test are available in the literature for this problem, very few procedures focuses on the distributional structure when the symmetry assumption is rejected.

Yanagimoto and Sibuya (1972) provided the various types of asymmetric distributional structure. We called them "*Positive biasedness.*" We list them below. Let F be the distribution function of random variable X. Note that $F(x-) = Pr[X < x]$.

Type 0 $\quad 1 - F(0) \geq F(0-)$,

Type I $\quad F(x) + F(-x-)$ for any $x \geq 0$,

Type II $\quad F(x+y) - F(x) \geq F(-x-) - F(-x-y-)$ for any $x, y > 0$,

Type III $\quad (F(x+y) - F(y))/(F(-y-) - F(-x-y-))$ is nondecreasing in both $x, y > 0$, and

Type IV $\quad (F(x+y) - F(y))/(F(-y-) - F(-x-y-))$ is nondecreasing in both $x > 0$ and y.

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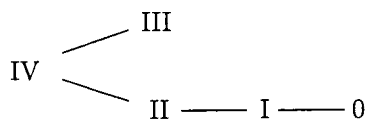
The implication scheme of positive biasedness is depicted as follows. This figure is adapted from Yanagomoto and Sibuya (1972).



Among these, Type II bias has received a substantial amount of interest because it is closely related to stochastic ordering. Specifically we say that X is positively type II biased if X is stochastically larger than $-X$. Dykstra, Kochar and Robertson (1995) studied likelihood ratio tests against type I and II positive biasedness. To our best knowledge no test for symmetry against type III or IV bias has given so far. This seems to be mainly due to that the restrictions of type III or IV positive biasedness are too strong.

In this paper we are going to consider the test of symmetry against several new positive biasedness restrictions which are stronger than type II but weaker than type IV (or type III). Among these restrictions one is related to uniform stochastic ordering and another one is related to likelihood ratio ordering. In section 2 we discussed these new types of positive biasedness and their relationship to various type of stochastic orderings. In section 3 estimation of distribution functions under new positive biasedness and likelihood ratio test for symmetry against new positive biasedness under discrete setting are discussed.

2. Near Type III Positive Bias

Let $\mathbf{p} = (p_{-k}, p_{-k+1}, \cdots, p_{-1}, p_0, p_1, \cdots, p_k)$ be $2k+1$ dimensional probability vector, i.e., $p_i > 0$ and $\sum_{i=-k}^{k} p_i = 1$. Then the type III bias can be expressed as $\sum_{j=i_1}^{i_2} p_j / \sum_{j=i_1}^{i_2} p_{-j}$ is nondecreasing in both $0 < i_1 \leq i_2$. The type IV bias is expressed as $\sum_{j=i_1}^{i_2} p_j / \sum_{j=i_1}^{i_2} p_{-j}$ is nondecreasing in both $i_1 \leq i_2$. It is clear that type III bias does not satisfy the restriction $\sum_{j=1}^{k} p_j / \sum_{j=1}^{k} p_{-j} \geq 1$ and hence does not imply type II nor type I bias. Now we add this restriction to type III biasedness. Then we call it *Near Type III Positive Bias.* It is easy to show that the near type III bias now imply type II and type I bias. Since $\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j}$ is nondecreasing in i_1 for type IV bias, i.e., $\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j} \geq 1$ for $i_1 = -k, \ldots, k$, then the type IV bias imply the near type III bias. Now we have new implication scheme as shown below.

IV —— Near III —— II —— I —— 0

The near type III restriction is, however, still too strong. Now we are going to pick up the two special cases from near type III positive biasedness. First let $i_1 > 0$ and $i_2 = i_1 + 1$. Then this restriction becomes

$$\frac{p_{i_1}}{p_{-i_1}} \text{ is nondecreasing in } i_1 = 1, \ldots, k, \text{ and } \sum_{j=1}^{k} p_j \geq \sum_{j=1}^{k} p_{-j}. \tag{2.1}$$

Second let $i_1 > 0$ and $i_2 = k$. Then it becomes

$$\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j} \text{ is nondecreasing in } i_1 = 1, \ldots, k, \text{ and } \sum_{j=1}^{k} p_j \geq \sum_{j=1}^{k} p_{-j}. \tag{2.2}$$

For convenience we call the former restriction a near type III(a) biasedness and the latter near type III(b) biasedness. It is not difficult to show that we have new implication scheme as follows.

IV —— Near III —— Near III(a) —— Near III(b) —— II —— I —— 0

Next we are going to find likelihood ratio tests of symmetry against near type III (a) and (b) positive biasedness.

3. Likelihood Ratio tests

For type I and II biasedness, Dykstra, Kochar and Robertson (1995b) proposed the likelihood ratio test statistic whose limiting null distribution is chi-bar-square distribution. In this section we consider the problem of testing the null hypothesis of symmetry about 0

$$H_0 \ : \ p_i = p_{-i} \text{ for } i = 1, \ldots, k$$

against the alternatives $H_1 - H_0$ and $H_2 - H_0$, where H_1 and H_2 postulate the restrictions (2.1) and (2.2) respectively.

3.1 Test of H_0 vs $H_1 - H_0$

First consider type III(a) restriction. We need to find the estimate of distribution function under type III(a) restriction, which can be achieved by maximizing

$$p_0^{n_0} \prod_{i=1}^{k} p_i^{n_i} p_{-i}^{n_{-i}} \tag{3.1}$$

subject to restriction (2.1), where n_i denote the the number of observations at $i = -k, \ldots, k$. We use a one-to-one transformation of parameter space. Let $A_1 = \sum_{i=1}^k p_i$, $A_{-1} = \sum_{i=1}^k p_{-i}$, $A_0 = p_0$, $a_i = p_i / A_1$, $a_{-i} = p_{-i}/A_{-1}$ for $i = 1, \ldots, k$, and $a_0 = 1$. Then we need to maximize

$$\prod_{i=1}^{k} a_i^{n_i} a_{-i}^{n_{-i}} \cdot A_1^{\sum_{i=1}^k n_i} A_{-1}^{\sum_{i=1}^k n_{-i}} A_0^{n_0} \tag{3.2}$$

subject to

$$\frac{a_i}{a_{-i}} \text{ is nondecreasing in } i = 1, \ldots, k, \text{ and } A_1 \geq A_{-1}, \tag{3.3}$$

and $a_i > 0$, $\sum_{i=1}^k a_i = \sum_{i=1}^k a_{-i} = 1$, and $\sum_{i=-1}^1 A_i = 1$. We note that no restrictions relate a_i's and A_i's to each other. This means that we only need to maximize (3.2) by maximizing two parts separately under corresponding restrictions. The former part is likelihood ratio ordering problem which was studied extensively by Dykstra, Kochar and Robertson(1995a). We use another one-to-one transformation again. Let $n_+ = \sum_{i=1}^k n_i$ and $n_- = \sum_{i=1}^k n_{-i}$. Let $\theta_i = n_+ a_i/(n_+ a_i + n_- a_{-i})$ and $\phi_i = n_+ a_i + n_- a_{-i}$. Then the maximization problem becomes

$$\text{maximize} \ \left(\frac{1}{n_+}\right)^{n_+} \left(\frac{1}{n_-}\right)^{n_-} \prod_{i=1}^{k} \theta_i^{n_i} (1-\theta_i)^{n_{-i}} \prod_{i=1}^{k} \phi_i^{n_i + n_{-i}} \tag{3.4}$$

subject to

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_k, \tag{3.5}$$

together with (a) $0 \leq \theta_i \leq 1$, $\phi_i \geq 0$ for $i = 1, \ldots, k$, (b) $\sum_{i=1}^k \phi_i = n_+ + n_-$, and (c) $\sum_{i=1}^k \theta_i \phi_i = n_+$. From Theorem 2.1 of Dykstra *et al.* (1995a) the constraint estimator of θ under (3.5) is given by $\boldsymbol{\theta}^* = E_{\mathbf{n}_+ + \mathbf{n}_-}(\hat{\theta}|I)$, where $\mathbf{n}_+ = (n_1, n_2, \ldots, n_k)$, $\mathbf{n}_- = (n_{-1}, n_{-2}, \ldots, n_{-k})$, $I = \{\mathbf{x} \in R^k : x_1 \leq x_2 \leq \cdots \leq x_k\}$, and $E_{\mathbf{w}}(\mathbf{x}|A)$ is the isotonic regression of $\mathbf{x}$ with respect to $\mathbf{w}$ onto A. All the vector operations are componentwise. Hence we have

$$\begin{aligned} a_i^* &= \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-} | I \right)_i, \\ a_{-i}^* &= \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-} | A \right)_i, \end{aligned}$$

for $i = 1, \ldots, k$, where $A = \{\mathbf{x} \in R^k : -\mathbf{x} \in I\}$.

Next we find the constraint estimator of A_i's under (3.3). Let $\mathbf{n}_A = (\sum_{i=1}^k n_{-i}, n_0, \sum_{i=1}^k n_i)$. Let $D = \{\mathbf{x} \in R^3 : x_1 \leq x_3\}$ and $n = n_- + n_0 + n_+$. Then

$$\mathbf{A}^* = (A_{-1}^*, A_0^*, A_1^*) = E\left(\frac{\mathbf{n}_A}{n} | D\right). \tag{3.6}$$

Theorem 3.1 If $n_i > 0$, then the maximum likelihood estimate of $\mathbf{p}$ under (2.1) is given by $\mathbf{p}^*$ where $\mathbf{p}^* = (p_{-k}^*, \ldots, p_k^*)$ with

$$\begin{aligned} p_i^* &= \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-} | I \right)_i \cdot E\left(\frac{\mathbf{n}_A}{n} | D\right)_1, \\ p_0^* &= E\left(\frac{\mathbf{n}_A}{n} | D\right)_0 = \frac{n_0}{n}, \\ p_{-i}^* &= \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-} | A \right)_i \cdot E\left(\frac{\mathbf{n}_A}{n} | D\right)_{-1}, \end{aligned}$$

for $i = 1, \ldots, k$.

Let $\mathbf{p}^\circ$ denote the estimate of $\mathbf{p}$ under null hypothesis. Then

$$\begin{aligned} p_i^\circ &= \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-} | C \right)_i \cdot E\left(\frac{\mathbf{n}_A}{n} | D^\circ\right)_1, \\ p_0^\circ &= E\left(\frac{\mathbf{n}_A}{n} | D^\circ\right)_0 = \frac{n_0}{n}, \\ p_{-i}^\circ &= \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-} | C \right)_i \cdot E\left(\frac{\mathbf{n}_A}{n} | D^\circ\right)_{-1}, \end{aligned}$$

for $i = 1, \ldots, k$, where $C = \{\mathbf{x} \in R^k : x_1 = x_2 = \cdots = x_k\}$, and $D^\circ = \{\mathbf{x} \in R^3 : x_1 = x_3\}$.

For discrete setting the restricted estimators are strongly consistent. This is due to the continuity property of isotonic regression with respect to weights and arguments. See Robertson *et al.* (1988).

The likelihood ratio test rejects H_0 in favor of H_1 for the large values of

$$\begin{aligned} T_{01} &= 2\left[\sum_{i=1}^k n_i(\ln\theta_i^* - \ln\theta_i^\circ) + \sum_{i=1}^k n_{-i}(\ln(1-\theta_i^*) - \ln(1-\theta_i^\circ))\right] \\ &\quad + 2\left[(\sum_{i=1}^k n_{-i})(\ln A_{-1}^* - \ln A_{-1}^\circ) + (\sum_{i=1}^k n_i)(\ln A_1^* - \ln A_1^\circ)\right]. \end{aligned}$$

Note that $A_0^\circ = A_0^*$. The following theorem provide the asymptotic null distribution of T_{01}. We omit the proof.

Theorem 3.2 If $p_i = p_{-i}$ for $i = i, \ldots, k$ and n goes to infinity, then for every $t > 0$,

$$\begin{aligned}\lim_{n\to\infty} Pr[T_{01} \geq t] &= \sum_{\ell=0}^{k} \frac{1}{2}\left[P(\ell, k; \mathbf{p}_+) + P(\ell+1, k; \mathbf{p}_+)\right] Pr[\chi_\ell^2 \geq t], \\ &\leq \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{1}{2}\right)^k Pr[\chi_\ell^2 \geq t],\end{aligned}$$

where $\mathbf{p}_+ = (p_1, \ldots, p_k)$, and $P(\ell, k; \mathbf{p}_+)$ for $\ell = 1, \ldots, k$ is the probability that $E_{\mathbf{p}_+}(\mathbf{X}|I)$ takes ℓ distinct values, where $\mathbf{X} = (X_1, \ldots, X_k)$ consists of independent random variables and X_i is $N(0, 1/p_i)$, and $P(0, k; \mathbf{p}_+) = P(k+1, k; \mathbf{p}_+) = 0$.

3.2 Test of H_0 vs $H_2 - H_0$

To find maximum likelihood estimate of $\mathbf{p}$ we consider the maximization of (3.1) under (2.2). We use the same reparametrization scheme as in (3.2). Restriction (2.2) becomes

$$\frac{\sum_{j=i}^k a_j}{\sum_{j=i}^k a_{-j}} \text{ is nondecreasing in } i = 1, \ldots, k, \text{ and } A_1 \geq A_{-1}. \tag{3.7}$$

The constraint on a_i and a_{-i} in (3.7) is uniform stochastic ordering. Dykstra *et al.* (1991) first proposed the statistical inference including likelihood ratio test for discrete distribution under uniform stochastic ordering. Using the same estimation procedure we can easily find the maximum likelihood estimate of a_I's under (3.7).

Let $\eta_i = \sum_{j=i+1}^k a_j / \sum_{j=i}^k a_j$ and $\eta_{-i} = \sum_{j=i+1}^k a_{-j} / \sum_{j=i}^k a_{-j}$. Then (3.7) becomes

$$\eta_{-i} \leq \eta_i \text{ for } i = 1, \ldots, k-1, \text{ and } A_1 \geq A_{-1}. \tag{3.8}$$

Now we need to find η and A which maximize

$$\prod_{i=1}^{k-1} \eta_i^{\sum_{j=i+1}^k n_j} (1-\eta_i)^{n_i} \eta_{-i}^{\sum_{j=i+1}^k n_{-j}} (1-\eta_{-i})^{n_{-i}} A_1^{\sum_{i=1}^k n_i} A_{-1}^{\sum_{i=1}^k n_{-i}} A_0^{n_0}$$

subject to (3.8). Note that no restriction relate the pairs (η_i, η_{-i}) for different values of i nor η's and A's. The constrained estimate of $\mathbf{A}$ is given in (3.6). Let $\mathbf{n}^{(i)} = (\sum_{j=i}^k n_j, \sum_{j=i}^k n_{-j})$. The constrained estimates of (η_i, η_{-i}) under (3.8) is given by $(\eta_i^\dagger, \eta_{-i}^\dagger)$ where, for $i = 1, \ldots, k-1$, $(\eta_i^\dagger, \eta_{-i}^\dagger) = E_{\mathbf{n}^{(i)}}\left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}}|I_2\right)$, where $I_2 = \{\mathbf{x} \in R^2 : x_1 \geq x_2\}$.

Theorem 3.3 If $n_i > 0$, then the maximum likelihood estimate of $\mathbf{p}$ under (2.2) is given by $\mathbf{p}^\dagger$ where $\mathbf{p}^\dagger = (p_{-k}^\dagger, \ldots, p_k^\dagger)$ with

$$p_i^\dagger = \begin{cases} (1 - E_{\mathbf{n}^{(i)}}(\mathbf{n}^{(i+1)}/\mathbf{n}^{(i)}|I_2)_1) \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_1, & \text{if } i = 1, \\ \prod_{j=1}^{i-1} E_{\mathbf{n}^{(j)}}(\mathbf{n}^{(j+1)}/\mathbf{n}^{(j)}|I_2)_1 & \\ \quad \cdot(1 - E_{\mathbf{n}^{(i)}}(\mathbf{n}^{(i+1)}/\mathbf{n}^{(i)}|I_2)_1) \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_1, & \text{if } i = 2, \ldots, k-1, \\ \prod_{j=1}^{k-1} E_{\mathbf{n}^{(j)}}(\mathbf{n}^{(j+1)}/\mathbf{n}^{(j)}|I_2)_1 \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_1, & \text{if } i = k, \end{cases}$$

$$p_0^\dagger = E\left(\frac{\mathbf{n}_A}{n}|D\right)_0 = \frac{n_0}{n},$$

$$p^{\dagger}_{-i} = \begin{cases} (1 - E_{\mathbf{n}^{(i)}}(\mathbf{n}^{(i+1)}/\mathbf{n}^{(i)}|I_2)_2) \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_{-1}, & \text{if } i = 1, \\ \prod_{j=1}^{i-1} E_{\mathbf{n}^{(j)}}(\mathbf{n}^{(j+1)}/\mathbf{n}^{(j)}|I_2)_2 & \\ \quad \cdot (1 - E_{\mathbf{n}^{(i)}}(\mathbf{n}^{(i+1)}/\mathbf{n}^{(i)}|I_2)_2) \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_{-1}, & \text{if } i = 2, \ldots, k-1, \\ \prod_{j=1}^{k-1} E_{\mathbf{n}^{(j)}}(\mathbf{n}^{(j+1)}/\mathbf{n}^{(j)}|I_2)_2 \cdot E\left(\frac{\mathbf{n}_A}{n}|D\right)_{-1}, & \text{if } i = k, \end{cases}$$

The likelihood ratio test rejects H_0 in favor of H_2 for large values of

$$\begin{aligned} T_{02} &= 2\left[\sum_{i=1}^{k-1}\left(\sum_{j=i+1}^{k} n_j\right)(\ln\eta_i^{\dagger} - \ln\eta_i^{\circ}) + \sum_{i=1}^{k-1} n_i(\ln(1-\eta_i^{\dagger}) - \ln(1-\eta_i^{\circ}))\right. \\ &\quad \left. + \sum_{i=1}^{k-1}\left(\sum_{j=i+1}^{k} n_{-j}\right)(\ln\eta_{-i}^{\dagger} - \ln\eta_{-i}^{\circ}) + \sum_{i=1}^{k-1} n_{-i}(\ln(1-\eta_{-i}^{\dagger}) - \ln(1-\eta_{-i}^{\circ}))\right] \\ &\quad + 2\left[\left(\sum_{i=1}^{k} n_{-i}\right)(\ln A_{-1}^{*} - \ln A_{-1}^{\circ}) + \left(\sum_{i=1}^{k} n_i\right)(\ln A_1^{*} - \ln A_1^{\circ})\right]. \end{aligned}$$

We state the following theorem without proof for asymptotic null distribution of T_{02}.

Theorem 3.4 If $p_i = p_{-i}$ for $i = 1, \ldots, k$ and n goes to infinity, then for every $t > 0$,

$$\lim_{n\to\infty} Pr[T_{02} \geq t] = \sum_{\ell=0}^{k} \binom{k}{\ell}\left(\frac{1}{2}\right)^k Pr[\chi_\ell^2 \geq t].$$

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