Convergence Analysis of LU Scheme for the Euler Equations on Unstructured Meshes

Joo Sung Kim[1] and Oh Joon Kwon [2]

1.Doctoral Candidate, Department of Aerospace Engineering, KAIST, soo@kaist.ac.kr
2. Associate Professor, Department of Aerospace Engineering, KAIST, ojkwon@kaist.ac.kr

Corresponding author ***Oh Joon Kwon***

Abstract

The convergence characteristics of the LU scheme for the Euler equations have been investigated by using the Von Neumann stability analysis. The results indicated that the convergence rate is governed by a specific combination of CFD parameters. Based on this insight, it is shown that the convergence characteristics of the LU scheme is not deteriorated at any grid aspect-ratio as long as the local time step is defined based on the parameter combination. The numerical results demonstrated that this time step definition provide a uniform convergence for grid aspect-ratios between one to 1×10^4.

Keyword: *LU scheme, Convergence analysis, Euler equations, Unstructured meshes*

1. Introduction

Most of the implicit schemes used in computational fluid dynamics (CFD) rely on some type of approximate inversion methods for solving the linear system of equations resulting from the local time linearization of the governing equations. Currently, several popular approximate inversion methods are available including the alternating-direction implicit (ADI) scheme, the line Gauss-Seidel (LGS) scheme, and the lower-upper (LU) scheme. The LU scheme was initially proposed by Jameson and Turkel and has been widely used over the past couple of decades. This scheme is particularly efficient for unstructured mesh topologies since it does not require any directional splitting in spatial coordinates.

For the Euler equations, there exist five independent CFD parameters which govern the convergence characteristics of the LU scheme: grid aspect-ratio, flow angle, Mach number, CFL number, and number of sub-iteration. In the past, several researches have been made to investigate the characteristics of the LU scheme. However, only one or two parameters have been considered in those studies. In the present study, the characteristics of the LU scheme for the Euler equations are assessed for all parameters excluding Mach number by using the Von Neumann stability analysis. Validation is made for a simple free stream flow with a point disturbance.

2. Stability Analysis

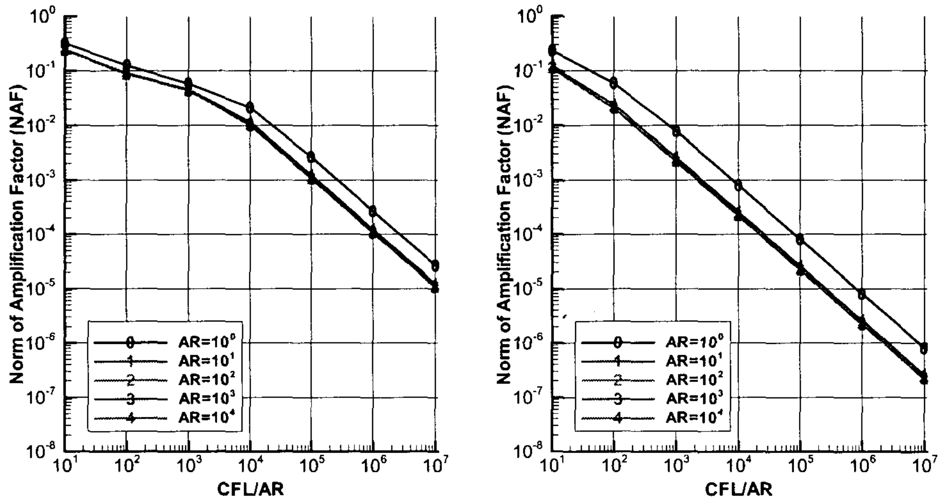
the multi-sweep LU scheme can be written as the following form:

$$(D+T_1+T_2)(Q^{k+1}-Q^n)-T_1D^{-1}T_2(Q^{k+1}-Q^k)=-R_i^n \qquad (1)$$

In the above equation, the first term on the left hand side corresponds to the direct inversion scheme. Also, the second term represents the approximate factorization (AF) error of the LU scheme. Therefore, for the complete analysis of the LU scheme, the contribution from each of the direct inversion and the AF error needs to be examined separately.

3. Direct Inversion Scheme

The norm of amplification factor (NAF) for the direct inversion scheme at a Mach number of 0.5 is shown in Fig. 1. Since the meaningful results can be obtained for the same value of θAR, each figure in Fig. 1 is calculated for a constant value of θAR: $\theta AR=10^{-2}$ in Fig. 1 (a) and $\theta AR=10^{0}$ in Fig. 1 (b). It is shown that the values of the NAF are approximately same for all aspect ratios of the grid for a fixed value of CFL/AR, irrespective of flow angles. Therefore, it can be concluded that the direct inversion scheme of the Euler equations does not suffer any convergence deterioration for any grid aspect-ratio as long as the CFL number is chosen properly from given CFL/AR.



(a) $\theta AR=10^{-2}$ **(b) $\theta AR=10^{0}$**

Fig. 1. Norms of Amplification Factor (NAF) of the Euler equations for $\theta AR=10^{-2}$ and $\theta AR=10^{0}$.

4. AF Error

Conceptually, the AF error may be regarded as the difference between the direct inversion and the LU scheme. Thus, the norm of the amplification factor defined as the difference can be used to examine the AF error:

$$NAF-AF=\left\{\frac{1}{2\,nmax}\sum_{i=1}^{nmax}\sum_{j=1}^{nmax}\left[\left|G_{ij,direct}(CFL,AR,\theta)\right|-\left|G_{ij,LU}(CFL,AR,\theta)\right|\right]^{2}\right\}^{1/2} \tag{2}$$

where $G_{ij,direct}$ and $G_{ij,LU}$ are the amplification factors of the direct inversion and LU schemes, respectively. As the value of $NAF\text{-}AF$ decreases to machine zero, the AF error is reduced and the performance of the LU scheme approaches to that of the direct inversion scheme.

Like the convergence rate of the direct inversion scheme, which is governed by a single parameter, CFL/AR, irrespective of the magnitude of the flow angle as shown in preceding section, the AF error also has a similar behavior. In Fig. 2, the $NAF\text{-}AF$ after the first sub-iteration is presented for $\theta AR=10^{0}$. The figure also indicates that the AF error remains unchanged even for a very large variation of the CFL number.

Next, the behavior of the AF error during subsequent sub-iterations is represented by the spectral radius to indicate the relative convergence rate.

$$Spectral\ Radius=\lim_{k\to\infty}\left(\frac{NAF-AF\ at\ sub-iteration=k}{NAF-AF\ at\ sub-iteration=1}\right)^{1/k-1} \tag{3}$$

In Fig. 3, the spectral radii of the NAF-AF are presented. The results show that the spectral radii are similar for all aspect ratios of the grid at a fixed value of *CFL/AR*. Thus, the magnitude of the AF error can be independent of grid if the *CFL* number is chosen by considering different grid aspect-ratios for given *CFL/AR*.

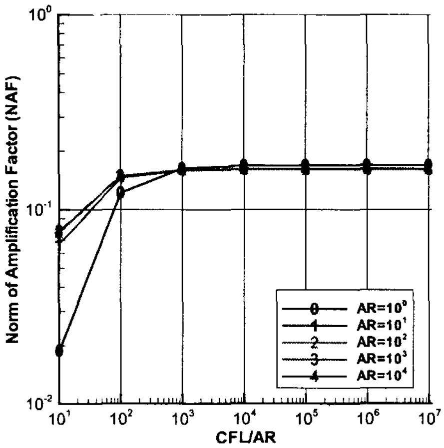


Fig. 2. Norm of Amplification Factor of the AF error for the Euler equations after the first sub-iteration with θAR=10^0.

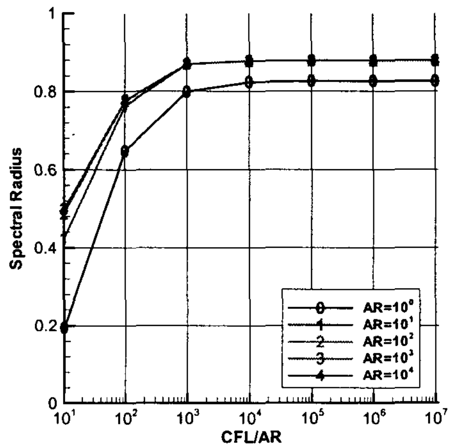


Fig. 3. Spectral Radius of the AF error for the Euler equations during sub-iteration with θAR=10^0.

4. Numerical Results

The results from the stability analysis are applied to a uniform flow in a rectangular domain by solving the Euler equations. The local grid aspect-ratio is arbitrarily changed from unity to 1×10^4 while maintaining a fixed grid size of 101×101. The initial condition used for all calculations is a uniform flow with 10% pressure perturbation at a point at the center of the domain. The freestream Mach number is 0.5.

In Fig. 4, the number of iterations required for the reduction of the error norm by 10 orders of magnitude is presented. It is shown that the convergence rate is approximately same for all aspect ratios of the grid for a fixed *CFL/AR*. It is also shown that the LU scheme does not suffer any convergence deterioration for any grid when large *CFL* numbers are used.

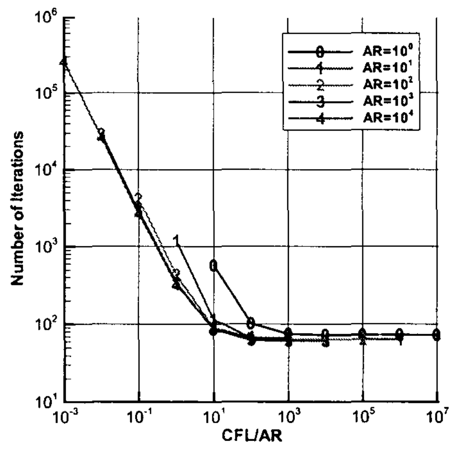


Fig. 4. Number of iterations to achieve 10 orders of error norm convergence.

5. Conclusions

A study has been made for the investigation of the convergence characteristics of the LU scheme for the Euler equations by using the Von Neumann stability analysis It was found that the convergence rate of the Euler equations is governed by a single parameter, *CFL/AR*, irrespective of the magnitude of the flow angle. This suggests that, when the local time step is defined based on *CFL/AR*, the LU scheme for the Euler equations does not suffer any convergence deterioration for any grid aspect-ratios. The numerical results demonstrate that this time step definition gives a uniform convergence for grid aspect-ratios from one to 1×10^4.