On the artificially-upstream flux splitting method

M. Sun and K. Takayama
Shock Wave Research Center, Institute of Fluid Science,
Tohoku University, Katahira 2-1-1, Aoba, Sendai 980, Japan
sun@ceres.ifs.tohoku.ac.jp and takayama@ifs.tohoku.ac.jp

Corresponding author ***M. Sun***

Abstract

A simple method is proposed to split the flux vector of the Euler equations by introducing two artificial wave speeds. The direction of wave propagation can be adjusted by these two wave speeds. This idea greatly simplifies the upwinding, and leads to a new family of upwind schemes. Numerical flux function for multi-dimensional Euler equations is formulated for any grid system, structured or unstructured. A remarkable simplicity of the scheme is that it successfully achieves one-sided approximation for all waves without recourse to any matrix operation. Moreover, its accuracy is comparable with the exact Riemann solver. For 1-D Euler equations, the scheme actually surpasses the exact solver in avoiding expansion shocks without any additional entropy fix. The scheme can exactly resolve stationary contact discontinuities, and it is also freed of the carbuncle problem in multi-dimensional computations.

Keyword: *upwind scheme, Euler equations, Riemann solver, structured grid, unstructured grid.*

Extended abstract

Upwind numerical methods are to discretize hyperbolic equations according to the direction of wave propagation. There are basically two approaches for determining upwind directions, namely the flux vector splitting (FVS) approach and the Godunov approach[1].

The Godunov-type approach uses either exact or approximate Riemann solutions between two adjacent states to calculate the flux through the interface between them. Most Godunov-type schemes have proven to be robust and accurate for the 1-D Euler equations. For multi-dimensional Euler equations, the extension based on 1-D Riemann solver, which obviously neglects the shear wave that only exists in multi-dimensions, contains a large amount of empiricism and must therefore remain suspect, althoughh these schemes have been successfully applied to many practical problems. Quirk [2] reported that many Godunov-type schemes contain subtle flaws that can cause spurious solutions, all in two dimensions. Carbuncle instability is one of the flaws, and has been extensively investigated. It was noticed that the Riemann solvers that explicitly take into account the presence of a contact surface, suffer from the carbuncle instability, while the FVS schemes exhibit absence of the instability[3]

The FVS approach achieves upwinding by decomposing the flux vector into positive and negative components according to the sign of eigenvalues of the coefficient matrix. The identification of upwind directions is done with less effort than in the Godunov-type methods. The upwinding is also realized for multi-dimensional flows. However, as compared with Godunov-type schemes, the FVS results in poorer resolution of discontinuity, particularly contact discontinuity. Most upwind schemes, either Godunov-type or FVS methods, have difficulties in resolving the sonic point, and produce a spurious expansion shock there.

In this work, a new method is proposed to split the flux vector of the Euler equations by introducing two artificial wave speeds. The direction of wave propagation is adjusted by these two wave speeds. If they are set to be the fastest wave speeds in two opposite directions, the method leads to the HLL approximate Riemann solver devised by Harten, Lax and van Leer [4], which indicates that the HLL solver is a vector flux splitting scheme as well as a Godunov-type scheme. A more accurate scheme that resolves 1-D contact discontinuity is further proposed by carefully choosing two wave speeds so that the flux vector is split to two simple flux vectors. One flux vector comes with either non-negative or non-positive eigenvalues and is easily solved by one-side differencing. Another flux vector becomes a system of two waves and one or two or three stationary contact discontinuities depending on the dimension of the Euler equations, which is solved by the HLL solver enforced with an isentropic condition. Numerical flux function for multi-dimensional Euler equations is formulated for any grid system, structured or unstructured. A remarkable simplicity of the scheme is that it successfully achieves one-sided approximation for all waves without recourse to any matrix operation. Moreover, its accuracy is comparable with the exact Riemann solver. For 1-D Euler equations, the scheme actually surpasses the exact solver in avoiding expansion shocks without any additional entropy fix. The scheme can exactly resolve stationary contact discontinuities, and it is also freed of the carbuncle problem in multi-dimensional computations. The robustness of the scheme is shown in 1-D test cases designed by Toro [1], and other 2-D calculations.

It turns out that the present scheme based on the idea of artificially-upstream (AU) splitting surpasses, to our knowledge, all existing upwind schemes, in one or few points that follow

1. The scheme is simple. It realizes one-side differencing for all waves in the 1-D and multi-dimensional flows without recourse to any matrix operation.
2. The scheme resolves sonic points smoothly. It implicitly and non-linearly introduces artificial viscosity only to the wave that possibly creates expansion shocks, while leaving other waves untouched.
3. The scheme can resolve exactly 1-D contact discontinuities. In multi-dimensions, the scheme does not suffer from carbuncle instabilities, which trouble the Godunov-type schemes that may resolve well contact discontinuities.
4. The resolution and robustness of the scheme is, overall, comparable with the exact Riemann solver.

References

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