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Applications of Stokes Eigenfunctions to the Numerical Solutions of the Navier-Stokes Equations in Channels and Pipes

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Abstract

General classes of boundary-pressure-driven flows of incompressible Newtonian fluids in three-dimensional (3D) channels and in 3D pipes with known steady laminar realizations are investigated respectively. The characteristic physical and geometrical quantities of the flows are subsumed in the kinetic Reynolds number Re and a parameter ψ, which involves the energetic ratio and the directions of the boundary-driven part and the pressure-driven part of the laminar flow.
The solution of non-stationary dimension-free Navier-Stokes equations is sought in the form $\underline{\mathbf{u}} = \mathbf{u}_L + \mathbf{u}$, where $\mathbf{u}_L$ is the scaled laminar velocity and periodical conditions are prescribed for $\mathbf{u}$ in the unbounded directions. The objects of our numerical investigations are autonomous systems (S) of ordinary differential equations for the time-dependent coefficients of the spatial Stokes eigenfunction, where these systems (S) were received by application of the Galerkin-method to the dimension-free Navier-Stokes equations for $\mathbf{u}$.

Keywords: Navier-Stokes equations, Stokes eigenfunctions, Galerkin methods, transition to turbulence

1. Introduction

The channel flow and the pipe flow of incompressible Newtonian fluids are well known as old-established objects of preference in theoretical and applied fluid-dynamic research. Owing to the simple geometry of the domains - these flows are ideal qualified for studies to the transition to turbulence and for investigations to extract further deterministic features from a random, fine-grained turbulent flow.
It is the purpose of our investigations to check and to clarify the possibilities of using low-dimensional Galerkin spaces defined by Stokes eigenfunctions for fact finding of the mechanism of the transition to turbulence in the non-stationary 3D-Navier-Stokes equations (NSE). Particularly, our studies are targeted on the behaviour of such approximations in the vicinities of critical Reynolds numbers.
We explore a general class of scaled flows of incompressible Newtonian fluids in unbounded channels and in unbounded pipes in $\mathbf{R}^3$, which can be described by the sum of laminar boundary-driven Couette (angular momentum for pipes) flow $\mathbf{u}_{L,c}$, of laminar pressure-driven Poiseuille flow $\mathbf{u}_{L,p}$, and of a time-dependent part $\mathbf{u}$. The first and second addenda are used to define the kinetic Reynolds number and a weighting parameter ψ for the energetic ratio and the direction of action of the boundary- and pressure-driven parts from $\mathbf{u}_L$. Low-dimensional approximation spaces spanned by the subsystems of Stokes eigenfunctions with periodic conditions in the former unbounded directions are applied for the direct numerical study of systems of Galerkin equations.
The essential notations and governing equations supplemented with initial and boundary conditions are outlined in section 2. After convenient scaling we decompose the velocity fields $\underline{\mathbf{u}}(t, \mathbf{x})$ into the laminar flow $\mathbf{u}_L(\mathbf{x})$ and the remaining velocity $\mathbf{u}(t, \mathbf{x})$ (fulfilling homogeneous Dirichlet conditions on the boundary of the pipe: $\underline{\mathbf{u}} = \mathbf{u}_L + \mathbf{u}$. Additionally, periodic conditions in former unbounded directions are required for $\mathbf{u}$. The pressure is decomposed similarly. The kinetic Reynolds numbers and a weighting parameter ψ for the energetic ratio and the direction of action of the boundary- and pressure-driven parts of $\mathbf{u}_L$ are also defined in this section.
We explain the Stokes operator and the Galerkin approximation $\mathbf{u}_N := \sum_{j=1}^{N} g_j \mathbf{w}_j$ of the so-called weak solutions $\mathbf{u}$ of the NSE for the remaining velocities in section 3, where the $\{\mathbf{w}_j\}_{j=1}^{\infty}$ are the

Stokes eigenfunctions. The Galerkin equations (as an autonomous system of ordinary differential equations for the coefficients $g_j(t)$ of the eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^N$) are stated there.
Section 4 is devoted to the numerical method and the results. At first a fixed period $2 \cdot l = 2 \cdot 2.69$ was chosen for historical reasons (cf. [6]). With this choice is the dimension of the Galerkin-space $N(\lambda) = N(\lambda_{max})$ determined by a bound λ_{max} for the eigenvalues λ. λ_{max} is taken in such a way, that the Galerkin-space includes two significant modes for the modification of the mean velocity both for the pure Couette-flow and the pure Poiseuille-flow. For the calculation of the coefficients in forming the system (S) are used universalized tools of combined C- and MAPLE-routines together with implemented rules of general addition theorems in form of allocation-lists.
The corresponding systems of ordinary differential equations were solved numerically for several values of the parameters Re, ψ and a set of initial values $\{g_j(0)\}, j = 1, ..., N$ (small $\mathbf{u}_{N,(0)} := \sum_{j=1}^N g_j(0)\mathbf{w}_j$), where the kinetic energy $E(t) := \sum_{j=1}^N g_j^2(t)$ of the Galerkin approximations $\mathbf{u}_N$ was used as a measure of turbulence.

2. The Basic Notations and Equations

The non-stationary NSE describe the time evolution of an incompressible Newtonian fluid. We are interested on the channel flow and on the pipe flow, which means, that the fluid is filling after scaling with R in [m] an open unbounded domain Ω in $\mathbf{R}^3$:

$$\Omega \quad := \quad \{R^{-1}\mathbf{y} = \mathbf{x} = (x_1, x_2, x_3)^T \in \mathbf{R}^3 : \sqrt{x_2^2 + \delta^* x_3^2} < 1\},$$

where $2R$ is the thickness of the channel (with $\delta^* = 0$) resp. the diameter of the pipe (with $\delta^* = 1$). We use $v_{char} = R(c_c^2 + c_p^2)^{1/2}$ as the velocity scale, which results from the common known stationary laminar velocity fields $\mathbf{v}_L(R\mathbf{x})$:

$$\mathbf{v}_L(R\mathbf{x}) := Rc_c \cdot \underbrace{((1-\delta^*)x_2, -\delta^* x_3, \delta^* x_2)^T}_{=\mathbf{u}_{L,c}} + Rc_p \cdot \underbrace{\chi(1 - (x_2^2 + \delta^* x_3^2), 0, 0)^T}_{=\mathbf{u}_{L,p}} = \mathbf{v}_{L,c} + \mathbf{v}_{L,p},$$

where c_c, c_p are velocity parameters in [1/s] and χ is chosen in such a way that for $c_c = c_p$ the velocity fields $\mathbf{v}_{L,c}$ and $\mathbf{v}_{L,p}$ result the equal kinetic energy in a control volume.
We establish the parameter $\psi \in [0, 2\pi)$ for all $\mathbf{v}_L \neq \mathbf{0}$ as an indicator for the energetic ratio and the direction of action of the parts of the laminar velocity by:

$$\cos\psi := c_c(c_c^2 + c_p^2)^{-1/2}, \quad \sin\psi := c_p(c_c^2 + c_p^2)^{-1/2}$$

We explain the *kinetic* Reynolds number Re by $Re = Re(c_c, c_p) := \frac{R^2}{\nu}(c_c^2 + c_p^2)^{1/2}$, the time scale $\frac{R^2}{\nu}$ and the scale for the kinematic pressure as $\nu(c_c^2 + c_p^2)^{1/2}$; where ν denotesthe kinematic viscosity in $[m^2/s]$.
The velocities and pressures are handled by the use of splitting up formulas:

$$\underline{\mathbf{u}} = \mathbf{u}_L + \mathbf{u} = \cos\psi\, \mathbf{u}_{L,c} + \sin\psi\, \mathbf{u}_{L,p} + \mathbf{u} \quad \text{and}$$
$$\underline{p} = p_L + p = \cos\psi\, p_{L,c} + \sin\psi\, p_{L,p} + p \quad ,$$

where $\mathbf{u}_L$ and p_L are the scaled laminar fields.
Finally periodic conditions for $\mathbf{u}$ and p are required instead of conditions in infinity: **(P):**
$\mathbf{u}(t, x_1, x_2, x_3) = \mathbf{u}(t, x_1{+}2l, x_2, x_3{+}2l(1{-}\delta^*)), p(t, x_1, x_2, x_3) = p(t, x_1{+}2l, x_2, x_3{+}2l(1{-}\delta^*))$
The initial-boundary value problem for the unknowns $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ is given by:
Problem 1: We seek solutions $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ fulfilling:

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta_{\mathbf{x}}\mathbf{u} + Re(\sum_{j=1}^3 u^j D_j \mathbf{u} + \sum_{j=1}^3 u_L^j D_j \mathbf{u} + \sum_{j=1}^3 u^j D_j \mathbf{u}_L) + \nabla_{\mathbf{x}} p = \mathbf{0} \quad ,$$

$$(\text{with } D_j := \partial./\partial x_j, \quad j = 1, 2, 3), \qquad \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0 \quad \text{in} \quad (0, T) \times \Omega$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) = \underline{\mathbf{u}}(0, .) - \mathbf{u}_L(.)$$

$$\mathbf{u}(t, x_1, x_2, x_3)_{|(x_2)^2 + \delta^*(x_3)^2 = 1} = \mathbf{0}, \quad \text{and } \mathbf{(P)}$$

3. Explanation of the Galerkin-Approximations

We restrict the domain $\boldsymbol{\Omega}$ on the open bounded sub-domain T_l considering the presupposed periodic conditions **(P)**: $T_l := \{\mathbf{x} \in \boldsymbol{\Omega} : \mathbf{x} = (x_1, x_2, x_3)^T;\ |x_1|,\ (1-\delta^*)|x_3| < l\}$,

in which we suppose $l \geq 1$.
It is convenient to use suitable function spaces in the mathematical treatment of Problem 1. Let us call these spaces $\mathbf{S} \subset (L_2(T_l))^3$ and $\mathbf{S}^1 \subset (W_2^1(T_l))^3$ (cf. [1], [3], [7]). The advantage of these spaces is, that their elements are solenoidal fields und fulfill the boundary conditions in a general sense. One can use the eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$ of the Stokes operator $\mathbf{A}$ to span the spaces $\mathbf{S}$ and $\mathbf{S}^1$, where one can understand the Stokes operator as a stationary version of Problem 1 with $Re = 0$. Additionally we note that the Stokes operator is an operator with a pure point spectrum with real eigenvalues $\lambda_j > 0$ of finite multiplicity (cf. [3]-[5]). We use the associated eigenfunctions $\{\mathbf{w}_j(\mathbf{x})\}_{j=1}^{\infty}$ of the Stokes operator $\mathbf{A}$ (counted in multiplicity) for the explanation of the Galerkin-approximations.
Let $(.,.)$ be the scalar product on $(L_2(T_l))^3$. We denote the number of eigenvalues of $\mathbf{A}$ with $\lambda_j \leq \lambda$ by $N = N(\lambda)$ and define the Galerkin-space $\mathbf{M}_N$ as the span of $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_N\}$ in $\mathbf{S}$.
We explain the Galerkin-approximation $\mathbf{u}_N(t,\mathbf{x}) := P_N u(t,\mathbf{x}) = \sum_{j=1}^{N} g_j(t)\mathbf{w}_j(\mathbf{x})$ (, with $g_j(t) :=, (\mathbf{u}(t,\mathbf{x}), \mathbf{w}_j)$, $j = 1, ..., N$,) of the so-called weak solution u(t,**x**) of Problem 1 by:

Problem 2 Let the initial-value $\mathbf{u}_0$ be an element of $\mathbf{S}$. We seek a function $\mathbf{u}_N \in C^1(0, T, \mathbf{M}_N)$ such that: for all $j \in \{1, \ldots, N\}$

$$\frac{d}{dt}(\mathbf{u}_N, \mathbf{w}_j) + (\nabla\mathbf{u}_N, \nabla\mathbf{w}_j) + Re\{b(\mathbf{u}_N, \mathbf{u}_N, \mathbf{w}_j) + b(\mathbf{u}_L, \mathbf{u}_N, \mathbf{w}_j) + b(\mathbf{u}_N, \mathbf{u}_L, \mathbf{w}_j)\} = 0,$$

$$\mathbf{u}_N(0) = P_N \mathbf{u}_0\,,$$

where we have used the abbreviation $b(\mathbf{u}, \mathbf{q}, \mathbf{s}) := (\sum_{j=1}^{3} u^j D_j \mathbf{q}, \mathbf{s})$ for the trilinear-form, which is antisymmetric in relation to permutation of $\mathbf{q}$ and $\mathbf{s}$.
The Problem 2 is equivalent to the solution of the initial-value problem of an autonomous systems **(S)** of ordinary differential equations for the time-dependent coefficients $g_j(t)$ with the initial-values $g_j(0)$.

4. Numerical Experiments and Results

We have used Dormand-Price methods (DOPRI5) and Runge-Kutta-Fehlberg (RKF45) methods (cf. [2]) with step size control for the numerical solution of the systems (S).
The evaluation of our numerical investigations shows very good agreements with measurements for the transition from laminar to turbulent flows in the vicinity of the critical Reynolds numbers and satisfactory results for the mean velocities of the turbulent flow (even for small dimension of our approximation space). However, the agreement with other experimental data for the Reynolds stresses and the root-mean-square values of the fluctuating velocities is less satisfactory, but due to the small dimension of our approximation space most probably.
Finally, we note that the approximate kinematic pressures $p_N(t)$ are to calculate straightforward from the $\{g_j(t)\}, j = 1, ..., N$.

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