Parametric Inference in an Exponential Distribution

Changsoo Lee[1], Joongdae Kim[2]

Abstract

Inference for probability $P(Y<X)$ in two-parameter exponential distribution when the scale parameters are known or not will be considered.

1. Introduction

A two-parameter exponential distribution is given by

$$f(x;\mu,\sigma)=\frac{1}{\sigma}e^{-(x-\mu)/\mu}, \quad x>\mu, \text{ where } \sigma>0, \mu\in R, \tag{1.1}$$

it will be denoted $X \sim \mathrm{EXP}(\mu, \sigma)$.

It is very important for us to consider an exponential distribution in parametric inferences. Here we shall consider inference for $P(Y<X)$ in two parameter exponential distribution.

The probability that a Weibull random variable Y is less than another independent Weibull random variable X was considered(McCool(1991)). Many other authors have considered the probability $P(Y<X)$, where X and Y are independent random variables.

The problem of estimating and of drawing inferences about, the probability that a random variable Y is less than an independent random variable X, arises in a reliability. When Y represents the random value of a stress that a device will be subjected to in service and X represents the strength that varies from item to item in the population of devices, then the reliability R, i.e. the probability that a randomly selected device functions successfully, is equal to $P(Y<X)$. The same problem also arises in the context of statistical tolerance where represents, say, Y the diameter of a draft and X the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then $P(Y<X)$.

In biometrics, Y represents a patient's remaining years of life if treated with drug A and X represents the patient's remaining years when treated with drug B. If the choice of drug is left to the patient, person's deliberations will center on whether $P(Y<X)$ is less

[1]Department of Multimedia Engineering, Kyungwoon University, Kumi, 730-850, Korea.
e-mail : cslee@ikw.ac.kr

[2]Department of Computer Information, Andong Junior College, Andong, 760-300, Korea.
e-mail : jdkim@andong-c.ac.kr

than or greater than 1/2.

Here, we shall consider inferences on $P(Y<X)$ in two parameter exponential distribution when the scale parameters are known or not : point and interval estimations, and test a null hypothesis.

2. Inference on P(X<Y)

Let X and Y be independent two-parameter exponential random variables, $X \sim$ EXP (μ_x, σ_x) and $Y \sim$ EXP(μ_y, σ_y), respectively.

Then, $$P(X<Y) = \int\int_{\mu_y<y<x} f_Y(y;\mu_y,\sigma_y)\cdot f_X(x;\mu_x,\sigma_x)dx$$
$$= 1-\frac{e^{\delta/\sigma_y}}{1+\sigma_x/\sigma_y}, \quad \text{where } \delta=\mu_y-\mu_x. \tag{2.1}$$

where $f_X(x)$ and $f_Y(y)$ are the density functions of X and Y, respectively.

To consider inferences on $P(X<Y)$, assume $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent random samples from $X \sim \text{EXP}(\mu_x, \sigma_x)$ and $Y \sim \text{EXP}(\mu_y, \sigma_y)$, respectively.

Then the MLE $\hat{\delta}$ of δ is

$$\hat{\delta} = \widehat{\mu_y} - \widehat{\mu_x} = Y_{(1)} - X_{(1)}, \tag{2.2}$$

where $X_{(1)}$ and $Y_{(1)}$ are the first order statistics of $X_i's$ and $Y_i's$, respectively.

By the result of Johnson, etal.(1995),

Fact 1. (a) $X_{(1)}$ follows an exponential distribution with a location parameter μ_x and a scale parameter σ_x/m.

(b) If $X_1, X_2, \cdots, X_m$ are iid exponential distributions with a scale parameter σ and a location parameter μ, then $\sum_{i=1}^{m}(X_i - X_{(1)})$ follows a gamma distribution with a shape parameter $m-1$ and a scale parameter σ.

(c) If a random variable X follows a gamma distribution with a shape parameter α and a scale parameter σ, then $E(\frac{1}{X^k}) = \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)\beta^k}$, for $\alpha > k$.

From Fact 1(a), we can obtain the expectation and variance of $\hat{\delta}$:

$$E(\hat{\delta}) = \delta + \frac{\sigma_y}{n} - \frac{\sigma_x}{m} \quad \text{and} \quad Var(\hat{\delta}) = \frac{\sigma_x^2}{m^2} + \frac{\sigma_y^2}{n^2}. \tag{2.3}$$

Let $D \equiv Y_{(1)} - X_{(1)}$. Then we can obtain the density function of D :

$$f_D(d) = \begin{cases} \dfrac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{n}{\sigma_y}(d-\delta)}, & \text{if } d \geq \delta \\ \dfrac{mn}{n\sigma_x + m\sigma_y} e^{-\frac{m}{\sigma_x}(\delta-d)}, & \text{if } d < \delta \end{cases} \tag{2.4}$$

2-A. When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known

From the result (2.1),

$$R = P(X < Y) = 1 - \frac{1}{2} e^{\delta/\sigma_0}, \qquad \delta = \mu_y - \mu_x.$$

Then the probability depends on δ only, Because R is a monotone function in δ, inference on δ is equivalent to inference on R. We hereafter confine attention to the parameter δ (see McCool(1991)).

When the scale parameters $\sigma_x = \sigma_y = \sigma_0$ is known, let $T = D - \delta$. Then from the pdf (2.4) of D, we have the pdf of T :

$$f_T(t) = \begin{cases} \dfrac{m}{m+n} \cdot \dfrac{n}{\sigma_0} e^{\frac{-n}{\sigma_0}t}, & \text{if } t \geq 0 \\ \dfrac{n}{m+n} \cdot \dfrac{m}{\sigma_0} e^{\frac{m}{\sigma_0}t}, & \text{if } t < 0. \end{cases} \tag{2.5}$$

Based on a pivotal quantity T, we shall consider an $(1 - p_1 - p_2)100\%$ confidence interval of δ.

For a given $0 < p_1 < 1$, there exists an b_1 such that $p_1 = \int_{-\infty}^{b_1} \frac{n}{m+n} \cdot \frac{m}{\sigma_0} e^{\frac{m}{\sigma_0}t} dt$, and

hence
$$b_1 = -\frac{\sigma_0}{2m} \cdot \chi^2_{2,\, \frac{m+n}{n} p_1}, \tag{2.6}$$

where $\alpha \equiv \int_{\chi^2_{2,\alpha}}^{\infty} \chi^2_2(t) dt$, $\chi^2_2(t)$ is the pdf of chi-square distribution of df 2.

For another given $0 < p_2 < 1$, there exists an b_2 such that $p_2 = \int_{b_2}^{\infty} \frac{m}{m+n} \cdot \frac{n}{\sigma_0} e^{-\frac{n}{\sigma_0}t} dt$, and

hence
$$b_2 = \frac{\sigma_0}{2n} \cdot \chi^2_{2,\, \frac{m+n}{m} p_2}. \tag{2.7}$$

Therefore, $(Y_{(1)} - X_{(1)} - b_2,\ Y_{(1)} - X_{(1)} - b_1)$ is an $(1 - p_1 - p_2)100\%$ confidence interval of δ.

Next We wish to test the null hypothesis $H_0 : \mu_x = \mu_y$ against $H_1 : \mu_x \neq \mu_y$.
Let $\Theta = \{(\mu_x, \mu_y) \mid \mu_x \in R,\ \mu_y \in R\}$ and $\theta = (\mu_x, \mu_y)$.
Then the joint pdf of $(X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ is

$$L(\theta)=f_{\theta}(x,y)=\prod_{i=1}^{m}\frac{1}{\sigma_0}e^{-\frac{1}{\sigma_0}(x_i-\mu_x)}\cdot\prod_{i=1}^{n}\frac{1}{\sigma_0}e^{-\frac{1}{\sigma_0}(y_i-\mu_y)},\ \text{for all}\ \ x_i>\mu_x,\ \ y_i>\mu_y.$$

From the likelihood function, we can obtain the MLE's of μ_x and μ_y,

$$\widehat{\mu_x}=X_{(1)}\ \text{ and }\ \widehat{\mu_y}=Y_{(1)}.$$

If $\mu_x=\mu_y=\mu$, then the MLE of μ is

$$\hat{\mu}=\min(X_{(1)},Y_{(1)})=(Y_{(1)}+X_{(1)}-|Y_{(1)}-X_{(1)}|)/2\ .$$

From definition of a likelihood ratio test(Rohatgi(1976)), the likelihood ratio test function can be obtained:

$$\lambda(x,y)=\exp(-|D|(\frac{m}{2\sigma_0}+\frac{n}{2\sigma_0})+D(\frac{m}{2\sigma_0}-\frac{n}{2\sigma_0})),\quad \text{where}\ \ D=Y_{(1)}-X_{(1)}.$$

Therefore, $\lambda(x,y)<c$ is equivalent to $D<b_1$ or $D>b_2$. (2.8)

Under H_0 : $\mu_x=\mu_y$, i.e. $\delta=0$, we hold $T=D-\delta=D$, and hence, for given $0<\alpha<1$ we can find b_1 and b_2 of (2.8), through the results (2.6) and (2.7) if $p_1=p_2=\alpha/2$.

2-B. When the scale parameters $\sigma_x=\sigma_y=\sigma$ is unknown

First we wish to know whether two scale parameters are equal or not:
To test the null hypothesis H_0 : $\sigma_x=\sigma_y=\sigma$ against H_1 : $\sigma_x\neq\sigma_y$, $\mu_x\in R$, $\mu_y\in R$
Let $\Theta=\{(\sigma_x,\sigma_y,\mu_x,\mu_y)\mid\sigma_x>0,\ \sigma_y>0,\ \mu_x\in R,\ \mu_y\in R\}$ and $\theta=(\sigma_x,\sigma_y,\mu_x,\mu_y)$.
Then the joint pdf of $(X_1,\cdots,X_m,Y_1,\cdots,Y_n)$ is

$$L(\theta)=f_{\theta}(x,y)=\prod_{i=1}^{m}\frac{1}{\sigma_x}e^{-\frac{1}{\sigma_x}(x_i-\mu_x)}\cdot\prod_{i=1}^{n}\frac{1}{\sigma_y}e^{-\frac{1}{\sigma_y}(y_i-\mu_y)},\ \text{for all}\ \ x_i>\mu_x,\ \ y_i>\mu_y.$$

Differentiating with respect to σ_x and σ_y, we can obtain the MLE's

$$\widehat{\sigma_x}=\frac{1}{m}\sum_{i=1}^{m}X_i,\ \ \widehat{\sigma_y}=\frac{1}{n}\sum_{i=1}^{n}Y_i,\ \text{and}\ \ \widehat{\mu_x}=X_{(1)}\ \text{ and }\ \widehat{\mu_y}=Y_{(1)}.$$

If $\sigma_x=\sigma_y=\sigma$, then the MLE of σ is

$$\hat{\sigma}=\frac{1}{n+m}(\sum_{i=1}^{m}(X_i-\widehat{\mu_x})+\sum_{i=1}^{n}(Y_i-\widehat{\mu_y})). \tag{2.9}$$

From definition of a likelihood ratio test(Rohatgi(1976)), the likelihood ratio test function can be obtained :

$$\lambda(x,y)=\left(\frac{\widehat{\sigma_x}}{\hat{\sigma}}\right)^m\cdot\left(\frac{\widehat{\sigma_y}}{\hat{\sigma}}\right)^n=\left(\frac{m+n}{m}\right)^m\cdot\left(\frac{m+n}{n}\right)^n\cdot\left(\frac{1}{1+1/U}\right)^m\cdot\left(\frac{1}{1+U}\right)^n,$$

where $U\equiv\dfrac{\sum_{i=1}^{m}(X_i-X_{(1)})}{\sum_{i=1}^{n}(Y_i-Y_{(1)})}$.

Therefore, $\lambda(x,y)<c$ is equivalent to $U<u_1$ or $U>u_2$. (2.10)

From Fact 1(b) and the results of Rohatgi(1976), we have the followings;

Fact 2. (a) $Z\equiv\dfrac{2\sum_{i=1}^{m}(X_i-X_{(1)})}{\sigma_x}$ and $W\equiv\dfrac{2\sum_{i=1}^{m}(Y_i-Y_{(1)})}{\sigma_y}$ follows chi-square distribution with df's 2(m-1) and 2(n-1), respectively.
(b) The random variables Z and W are independent.

Under H_0 : $\sigma_x=\sigma_y=\sigma$, from Fact 2, $U\equiv\dfrac{\sum_{i=1}^{m}(X_i-X_{(1)})}{\sum_{i=1}^{n}(Y_i-Y_{(1)})}$ follows a F-distribution with df's 2(m-1) and 2(n-1). And hence, for a given $0<\alpha<1$,

$$u_2=F_{\alpha/2}(2(m-1),2(n-1)) \text{ and } u_1=1/F_{\alpha/2}(2(n-1),2(m-1)) \quad , \quad \text{from (2.10).}$$

If $\sigma_x=\sigma_y=\sigma$, then from the result(2.1),

$$R=P(X<Y)=1-\frac{1}{2}e^{\delta/\sigma}, \quad \text{where } \delta=\mu_y-\mu_x$$

Let $\beta\equiv\delta/\sigma$. Then, an estimator of β is defined as:

$$\hat{\beta}\equiv\hat{\delta}/\hat{\sigma}=\frac{(m+n)(Y_{(1)}-X_{(1)})}{\sum_{i=1}^{m}(X_i-X_{(1)})+\sum_{i=1}^{n}(Y_i-Y_{(1)})}, \quad \text{from results (2.2) and (2.9).}$$

From the results (2.3) and Fact 1(c), we can obtain the followings :

$$E(\hat{\beta})=\beta+\frac{3}{m+n-3}\beta+\frac{m^2-n^2}{mn(m+n-3)}$$

and

$$Var(\hat{\beta})=\frac{(m+n)^2(m^2+n^2)}{m^2n^2(m+n-3)^2(m+n-4)}.$$

References

Johnson, N. L., Kotz, S. and Balakrishnan, N.(1995), *Continuous Univariate Distributions-2,* John Wiley & Sons, New York.

McCool, J.I.(1991), Inference on P(X<Y) in the Weibull Case, *Communications in Statistics - Simulations*, 20(1), 129-148.

Rohatgi, V.K.(1976), *An Introduction to Probability Theory and Mathematical Statistics*, John Wiley & Sons, New York.