THE EXPANSION OF MEAN DISTANCE OF BROWNIAN MOTION ON RIEMANNIAN MANIFOLD*

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ABSTRACT

We study the asymptotic expansion in small time of the mean distance of Brownian motion on Riemannian manifolds. We compute the first four terms of the asymptotic expansion of the mean distance by using the decomposition of Laplacian into homogeneous components. This expansion can be expressed in terms of the scalar valued curvature invariants of order 2, 4, 6.

Keywords: Brownian motion; Riemannian manifold; Curvature invariants; Ricci curvature; Scalar curvature; Bianchi's identity; Ricci's identity

1. INTRODUCTION

Suppose that (M, g) is an n-dimensional connected Riemannian manifold and X_t is a Brownian motion on M starting at $m \in M$. Let $\gamma_t = d(X_t, m)$ be the radial part of a Brownian motion on M where d is the Riemannian distance induced by a Riemannian metric g.

Kim and Park(2002) improve the result of Liao and Zheng(1995) by using a method which consists in normal coordinates $(X^1, \cdots, X^n)$ in a neighborhood of m. Let $(X^1, \cdots, X^n)$ be a solution of the following SDE:

$$\begin{cases} dX_t^i = \sigma_{ik}(X_t)dB_t^k + a^i(X_t)dt \\ X_0 = 0, \end{cases} \tag{1.1}$$

where $(\sigma_{ik}(x))$ is the square root of $(g^{ik}(x))$ in the normal coordinates, $(B^1, \cdots, B^n)$ is an n-dimensional Brownian motion and $a^i(x)$ is given by $a^i(x) = -(1/2)g^{jk}(x)\Gamma^i_{jk}(x)$. By using the SDE and Taylor development of g, a and σ, Kim and Park(2002) obtain the asymptotic expansion of the mean distance up to order 3, that is,

$$\begin{aligned} & E[\gamma^2(X_t, m)] \\ = \; & nt - \frac{1}{6}\tau(m)t^2 - \frac{1}{90}\left(6\Delta\tau(m) + \|R(m)\|^2 - \|\rho(m)\|^2\right)t^3 + o(t^3) \quad \text{as} \quad t \downarrow 0. \end{aligned} \tag{1.2}$$

The previous methods, used by Kim and Park(2002), do not seem to be readily applicable to the expansion of higher order. If we use these methods, the computation of the coefficient of t^4 seems hopelessly complicated. In this paper we develop a new method to compute the coefficients in the power series expansion of $E(\gamma_t^2)$. Also this method turns out much simpler than the previous methods for the expansion up to order 3. For the calculation of

*This research was supported (in part) by KOSEF through Statistical Research Center for Complex Systems at Seoul National University.

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the coefficients of higher order of this expansion, we need to compute $\Delta^p\gamma^2(m)$, for $p \geq 3$. In Section 2, we calculate explicitly $\Delta^2\gamma(m)$, $\Delta^3\gamma(m)$ and $\Delta^4\gamma(m)$ by the decomposition of Laplacian into homogeneous components and give asymptotic expansion up to order 4 in small time. We are only interested in computing the coefficients of the power series expansion.

Now we introduce the several curvatures. We fix a normal coordinate system $(x^1, \ldots, x^n)$ in a neighborhood of the point m. Let g_{ij}, g^{ij} and Γ^i_{jk} be the components of Riemannian metric, the inverse and Christoffel symbol, respectively. R_{ijkl} is the components of the curvature tensor and ρ_{ij} is the components of the Ricci curvature, that is, $\rho_{ij} = \sum_k^n R_{kikj}$. Also $\tau = \sum_{i=1}^n \rho_{ii}$ is the scalar curvature.

By definition a scalar valued curvature invariant is a polynomial in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of basis of the tangent space T_mM. Such a scalar valued invariant is said to have order k if it has a term of k-th derivatives of the metric tensor. Let $I(k,n)$ be a space of invariants of order $2k$ for Riemannian manifolds of dimension n. Then $\dim I(1,n) = 1$ for $n \geq 2$ and $\dim I(2,n) = 4$ for $n \geq 4$. Let $||R|| = (\sum(R^i_{jkl})^2)^{1/2}$ and $||\rho|| = (\sum \rho^2_{ij})^{1/2}$ be the lengths of the curvature tensor and the Ricci curvature, respectively. Then $\{\tau\}$ is a basis for $I(1,n)$ and $\{\tau^2, ||\rho||^2, ||R||^2, \Delta\tau\}$ is a basis for $I(2,n)$. Also if $n \geq 6$, $I(3,n)$ has dimension 17. Using the same notations as those of Gray and Vanhecke(1979) we write a basis for $I(3,n)$:

$$\begin{aligned}
&\tau^3,\ \tau||\rho||^2,\ \tau||R||^2,\ \check{\rho} = \rho_{ab}\rho_{bc}\rho_{ca},\ <\rho, \dot{R}>= \rho_{ab}R_{apqr}R_{bpqr}, ||\nabla\tau||^2 = (\nabla_a\tau)^2,\\
&<\rho\otimes\rho,\ \bar{R}>= \rho_{ab}\rho_{cd}R_{acbd},\ \check{R} = R_{abcd}R_{cdpq}R_{pqab},\ \hat{R} = R_{acbd}R_{cpdq}R_{paqb},\\
&||\nabla\rho||^2 = (\nabla_a\rho_{bc})^2,\ \alpha(\rho) = \nabla_a\rho_{bc}\nabla_c\rho_{ab}, ||\nabla R||^2 = (\nabla_a R_{bcdq})^2,\ \tau\Delta\tau,\ \Delta^2\tau,\\
&<\Delta\rho, \rho>= \rho_{ab}\nabla^2_{cc}\rho_{ab},\ <\nabla^2\tau, \rho>= \nabla^2_{ab}\tau\rho_{ab},\ <\nabla R, R>= R_{abcd}\nabla^2_{pp}R_{abcd}.
\end{aligned} \tag{1.3}$$

Our main result is

$$\begin{aligned}
&E[\gamma^2(X_t,m)]\\
= \ & nt - \frac{1}{6}\tau(m)t^2 - \frac{1}{90}\Big(6\Delta\tau(m) - ||\rho||^2 + ||R||^2\Big)t^3\\
&+\frac{1}{840\times 24}\Big(- 270\Delta^2\tau - 45||\nabla\tau||^2 - 96 <\nabla^2\tau, \rho> +30||\nabla\rho||^2\\
&+148 <\Delta\rho, \rho> +60\alpha(\rho) + \frac{1268}{3}\check{\rho} - \frac{664}{3} <\rho\otimes\rho, \bar{R}> -180 <\Delta R, R>\\
&+\frac{296}{3} <\rho, \dot{R}> -\frac{400}{3}\hat{R} - \frac{220}{3}\check{R} - 135||\nabla R||^2\Big)t^4 + o(t^4) \quad \text{as} \quad t \downarrow 0.
\end{aligned} \tag{1.4}$$

In this paper we will use the summation convention, that is, we will omit the summation sign over repeated indices. Also we omit the argument $m \in M$ if there is no room for confusion, for example, $||R(m)|| = ||R||$.

2. PRELIMINARIES

By Gray and Pinsky(1983), for any smooth function f defined in a neighborhood of m, we have

$$\Delta f = \Delta_{-2}f + \sum_{h=0}^{\infty}\Delta_h f. \tag{2.1}$$

Here Δ_h is a second-order differential operator on M near m which maps k-th polynomials into $(k+h)$-th degree polynomials. We write the first few terms Δ_h. For any system of normal coordinates $(x_1, \ldots, x_n)$ at m which is identified with $0 = (0, \ldots, 0) \in R^n$;

$$\begin{aligned}
\Delta_{-2} &= \partial_i \partial_i, \\
\Delta_0 &= \frac{1}{3} R_{iajb} x_a x_b \partial_i \partial_j - \frac{2}{3} \rho_{ia} x_a \partial_i, \qquad (2.2) \\
\Delta_1 &= \frac{1}{6} \nabla_a R_{ibjc} x_a x_b x_c \partial_i \partial_j + \frac{1}{12} (\nabla_i \rho_{ab} - 6 \nabla_a \rho_{ib}) x_a x_b \partial_i, \\
\Delta_2 &= \frac{1}{60} (3 \nabla^2_{ab} R_{icjd} + 4 R_{aibp} R_{cjdp}) x_a x_b x_c x_d \partial_i \partial_j \\
&\quad + \frac{1}{180} (9 \nabla^2_{ai} \rho_{bc} - 36 \nabla^2_{ab} \rho_{ic} - 14 \rho_{ap} R_{pbic} - 16 R_{apbq} R_{ipcq}) x_a x_b x_c \partial_i, \qquad (2.3) \\
&\quad \vdots
\end{aligned}$$

In order to calculate the coefficients of t^2, t^3 and t^4, we need the decompositions of Δ^2, Δ^3 and Δ^4. It follows from (2.1) that

$$\Delta^2 f = \Delta^2_{-2} f + \sum_{h=0}^{\infty} \Delta_{-2} \Delta_h f + \sum_{l=0}^{\infty} \Delta_l (\Delta f), \qquad (2.4)$$

$$\begin{aligned}
\Delta^3 f &= \Delta^3_{-2} f + \sum_{h=0}^{\infty} \Delta^2_{-2} \Delta_h f + \sum_{l=0}^{\infty} \Delta_{-2} \Delta_l \Delta_{-2} f + \sum_{l=0}^{\infty} \sum_{h=0}^{\infty} \Delta_{-2} \Delta_l \Delta_h f \\
&\quad + \sum_{k=0}^{\infty} \Delta_k (\Delta^2 f), \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
\Delta^4 f &= \Delta^4_{-2} f + \sum_{h=0}^{\infty} \Delta^3_{-2} \Delta_h f + \sum_{l=0}^{\infty} \Delta^2_{-2} \Delta_l \Delta_{-2} f + \sum_{l=0}^{\infty} \sum_{h=0}^{\infty} \Delta^2_{-2} \Delta_l \Delta_h f \\
&\quad + \sum_{k=0}^{\infty} \Delta_{-2} \Delta_k \Delta^2_{-2} f + \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \Delta_{-2} \Delta_k \Delta_{-2} \Delta_h f + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Delta_{-2} \Delta_k \Delta_l \Delta_{-2} f \\
&\quad + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{h=0}^{\infty} \Delta_{-2} \Delta_k \Delta_l \Delta_h f + \sum_{g=0}^{\infty} \Delta_g (\Delta^3 f). \qquad (2.6)
\end{aligned}$$

The general formula Δ^p is given by

$$\begin{aligned}
\Delta^p &= \Delta^p_{-2} + \sum_{h=0}^{\infty} \sum_{s_k = h}' \Delta_{s_1} \cdots \Delta_{s_k} \cdots \Delta_{s_p} + \sum_{l,h=0}^{\infty} \sum_{s_k = h, s_r = l}' \Delta_{s_1} \cdots \Delta_{s_k} \cdots \Delta_{s_r} \cdots \Delta_{s_p} \\
&\quad + \cdots + \sum_{s_1, \ldots, s_p = 0}^{\infty} \Delta_{s_1} \Delta_{s_1} \cdots \Delta_{s_p},
\end{aligned}$$

where

$$\sum_{s_k = h}' \Delta_{s_1} \cdots \Delta_{s_k} \cdots \Delta_{s_p} = \Delta_h \Delta_{-2} \cdots \Delta_{-2} + \Delta_{-2} \Delta_h \Delta_{-2} \cdots \Delta_{-2} + \cdots + \Delta_{-2} \cdots \Delta_{-2} \Delta_h$$

$$\begin{aligned}
\sum_{s_k = h, s_r = l}' \Delta_{s_1} \cdots \Delta_{s_k} \cdots \Delta_{s_r} \cdots \Delta_{s_p} &= \Delta_l \Delta_h \Delta_{-2} \cdots \Delta_{-2} + \Delta_l \Delta_{-2} \Delta_h \Delta_{-2} \cdots \Delta_{-2} + \cdots \\
&\quad + \Delta_{-2} \cdots \Delta_{-2} \Delta_l \Delta_h, \\
&\quad \cdots\cdots
\end{aligned}$$

In particular, if $f(x) = d^2(x,m)$, then $\Delta_{-2} f = 2n$. From (2.1), we have

$$\Delta^2 f = \sum_{h=0}^{\infty} \Delta_{-2}\Delta_h f + \sum_{l=0}^{\infty} \Delta_l(\Delta f), \tag{2.7}$$

$$\Delta^3 f = \sum_{h=0}^{\infty} \Delta_{-2}^2\Delta_h f + \sum_{l=0}^{\infty}\sum_{h=0}^{\infty} \Delta_{-2}\Delta_l\Delta_h f + \sum_{k=0}^{\infty} \Delta_k(\Delta^2 f), \tag{2.8}$$

$$\begin{aligned}\Delta^4 f = & \sum_{h=0}^{\infty} \Delta_{-2}^3\Delta_h f + \sum_{l=0}^{\infty}\sum_{h=0}^{\infty} \Delta_{-2}^2\Delta_l\Delta_h f + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \Delta_{-2}\Delta_k\Delta_{-2}\Delta_h f \\ & + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{h=0}^{\infty} \Delta_{-2}\Delta_k\Delta_l\Delta_h f + \sum_{g=0}^{\infty} \Delta_g(\Delta^3 f).\end{aligned} \tag{2.9}$$

Using the above formulas, we compute $\Delta^p\gamma^2(m)$ for $p = 2, 3, 4$ in the next section.

3. GEOMETRIC LEMMAS

We give some formulas which are used to calculate $\Delta^2\gamma^2(m)$, $\Delta^3\gamma^2(m)$ and $\Delta^4\gamma^2(m)$. All of the formulas are the consequences of the symmetries of the curvature operator including the two Bianchi's identities and the Ricci's identity, which can be found in Gray and Vanhecke(1979), and also in Sakai(1971).

We establish the following Lemma for calculating $\Delta^2\gamma^2(m)$.

Lemma 3.1. *Let $\gamma(x) = d(x,m)$ for fixed $m \in M$. Then we have*

$$\Delta_{-2}\Delta_h\gamma^2(m) = \begin{cases} -\frac{4}{3}\tau(m) & \text{for } h = 0, \\ 0 & \text{for all } h \geq 1, \end{cases}$$

$$\Delta_l\Delta_h\gamma^2(m) = 0, \quad \text{for all } h, l \geq 0.$$

Now we compute the terms in $\Delta^3\gamma^2(m)$.

Lemma 3.2. *We have the following:*

$$\Delta_{-2}^2\Delta_h\gamma^2(m) = \begin{cases} \frac{8}{45}\Big(- 18\Delta\tau(m) - 2\|\rho\|^2 - 3\|R\|^2\Big) & \text{for } h = 2, \\ 0 & \text{for all } h \neq 2, \end{cases}$$

$$\Delta_{-2}\Delta_l\Delta_h\gamma^2(m) = \begin{cases} \frac{8}{9}\|\rho\|^2 & \text{for } l = 0 \text{ and } h = 0, \\ 0 & \text{for } l > 0 \text{ or } h > 0, \end{cases}$$

$$\Delta_l\Delta_{-2}\Delta_h\gamma^2(m) = 0 \quad \text{for all } l, h \geq 0,$$

$$\Delta_k\Delta_l\Delta_h\gamma^2(m) = 0 \quad \text{for all } k, l, h \geq 0.$$

Finally we compute the terms that are needed to calculate $\Delta^4\gamma^2(m)$.

Lemma 3.3. *We have the following formulas:*

$$\Delta_{-2}^{3}\Delta_h\gamma^2(m)=\begin{cases}\frac{2}{105}\Big(-270\Delta^2\tau-150\|\nabla\tau\|^2-432<\nabla^2\tau,\rho>-75\|\nabla\rho\|^2\\ +36<\Delta\rho,\rho>-150\alpha(\rho)-\frac{824}{3}\check{\rho}+264<\rho\otimes\rho,\bar{R}>\\ -180<\Delta R,R>+24<\rho,\dot{R}>-\frac{400}{3}\hat{R}-\frac{220}{3}\check{R}\\ -135\|\nabla R\|^2\Big)\quad for\ \ h=4,\\ 0\quad for\ all\ \ h\neq 4,\end{cases}$$

$$\Delta_{-2}^{2}\Delta_l\Delta_h\gamma(m)=\begin{cases}\frac{8}{135}\Big(54<\nabla^2\tau,\rho>-64<\rho\otimes\rho,\bar{R}>\\ +72\check{\rho}+12<\rho,\dot{R}>+18<\Delta\rho,\rho>\Big)\quad for\ \ l=0,h=2,\\ \frac{8}{135}\Big(27<\nabla^2\tau,\rho>-70<\rho\otimes\rho,\bar{R}>\\ +134\check{\rho}+6<\rho,\dot{R}>+9<\Delta\rho,\rho>\Big)\quad for\ \ l=2,h=0,\\ \frac{2}{3}\Big(6\alpha(\rho)+3\|\nabla\rho\|^2+2\|\nabla\tau\|^2\Big)\quad for\ \ l=1,h=1,\\ 0\quad for\ \ l+h\neq 2,\end{cases}$$

$$\Delta_{-2}\Delta_l\Delta_{-2}\Delta_h\gamma(m)=\begin{cases}\frac{8}{135}\Big(9<\Delta\rho,\rho>-32<\rho\otimes\rho,\bar{R}>+27<\nabla^2\tau,\rho>\\ +36\check{\rho}+6<\rho,\dot{R}>\Big)\quad for\ \ l=0,h=2,\\ \frac{2}{3}\|\nabla\tau\|^2\quad for\ \ l=1,h=1,\\ 0\quad otherwise,\end{cases}$$

$$\Delta_{-2}\Delta_k\Delta_l\Delta_h\gamma(m)=\begin{cases}\frac{16}{27}(<\rho\otimes\rho,\bar{R}>-2\check{\rho})\quad for\ all\ \ k=l=h=0\\ 0\quad otherwise,\end{cases}$$

Using (2.7)-(2.9) and Lemma 3.1-3.3, we obtain the following Theorem.

Theorem 3.1. *For fixed $m\in M$, we have*

$$\begin{aligned}\Delta^2\gamma^2(m)&=-\frac{4}{3}\tau(m),\\ \Delta^3\gamma^2(m)&=-\frac{8}{15}\Big(6\Delta\tau(m)-\|\rho\|^2+\|R\|^2\Big),\\ \Delta^4\gamma^2(m)&=\frac{2}{105}\Big(-270\Delta^2\tau-45\|\nabla\tau\|^2-96<\nabla^2\tau,\rho>+30\|\nabla\rho\|^2\\ &\quad+148<\Delta\rho,\rho>+60\alpha(\rho)+\frac{1268}{3}\check{\rho}-\frac{664}{3}<\rho\otimes\rho,\bar{R}>-180<\Delta R,R>\\ &\quad+\frac{296}{3}<\rho,\dot{R}>-\frac{400}{3}\hat{R}-\frac{220}{3}\check{R}-135\|\nabla R\|^2\Big).\end{aligned}$$

4. THE MAIN THEOREM

Applying Ito's formula to $\gamma_t=d(X_t,m)$, we have

$$\gamma^2(X_{t\wedge T_\epsilon},m)=M_{t\wedge T_\epsilon}+\frac{1}{2}\int_0^t I_{\{s<T_\epsilon\}}\Delta\gamma^2(X_{s\wedge T_\epsilon},m)ds,$$

where M_t is a martingale. Hence

$$E[\gamma^2(X_{t\wedge T_\epsilon}, m)] \;=\; \frac{1}{2}\left\{ \int_0^t E[\Delta\gamma^2(X_{s\wedge T_\epsilon}, m)]ds - E[(t-(t\wedge T_\epsilon))\Delta\gamma^2(X_{T_\epsilon}, m)] \right\}.$$

Again applying Ito's formula to $\Delta\gamma_t^2 = \Delta\gamma^2(X_t, m)$, we obtain

$$E[\Delta\gamma^2(X_{s\wedge T_\epsilon}, m)] \;=\; 2n + \frac{1}{2}\left\{ \int_0^s E[\Delta^2\gamma^2(X_{u\wedge T_\epsilon}, m)]du - E[(s-(s\wedge T_\epsilon))\Delta^2\gamma^2(X_{T_\epsilon}, m)] \right\}.$$

By successive applications of Ito's formula to $\Delta^2\gamma_t^2 = \Delta^2\gamma^2(X_s, m)$ and $\Delta^3\gamma_t^2 = \Delta^3\gamma^2(X_u, m)$, we obtain $E[\Delta^2\gamma^2(X_{u\wedge T_\epsilon}, m)]$ and $E[\Delta^3\gamma^2(X_{v\wedge T_\epsilon}, m)]$. From this and Theorem 3.1, we have

$$\begin{aligned}
& E[\gamma^2(X_{t\wedge T_\epsilon}, m)] \\
= \;& nt - \frac{1}{6}\tau(m)t^2 - \frac{1}{90}\Big(6\Delta\tau(m) - ||\rho||^2 + ||R||^2\Big)t^3 \\
& + \frac{1}{840\times 24}\Big(-270\Delta^2\tau - 45||\nabla\tau||^2 - 96 < \nabla^2\tau, \rho > +30||\nabla\rho||^2 \\
& +148 < \Delta\rho, \rho > +60\alpha(\rho) + \frac{1268}{3}\check{\rho} - \frac{664}{3} < \rho\otimes\rho, \bar{R} > -180 < \Delta R, R > \\
& + \frac{296}{3} < \rho, \dot{R} > - \frac{400}{3}\hat{R} - \frac{220}{3}\check{R} - 135||\nabla R||^2\Big)t^4 + o(t^4). \qquad (4.1)
\end{aligned}$$

Under some global bounded conditions on M, we obtain (1.4) from (4.1).

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