An application of a short time expansion of the heat kernel by the generalized Wiener functional*

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Using a short time expansion of the fundamental solution of heat equation by analysis of Wiener functional with the help of Malliavin calculus, we obtain the asymptotic expansion of the mean distance of Brownian motion on Riemannian manifolds.

Key words and Phrases: Generalized Wiener functional, Riemannian manifold, Ornstein-Uhlenbeck operator, Scalar curvature, Malliavin calculus.

1 Introduction

Let (M, g) be an d-dimensional compact smooth Riemannian manifold and $X(t)$ be a Brownian motion on M starting at $p \in M$. Let $\gamma_t = d(X(t), p)$ be the radial part of a Brownian motion on M where d is the Riemannian distance induced by a Riemannian metric g. We interoduce the several curvatures. R_{ijkl} are the components of the curvature tensor and ρ_{ij} are the components of the Ricci curvature. Also $\tau = \sum_{i=1}^{d} \rho_{ii}$ is the scalar curvature. Let $\|R\|^2 = \sum(R_{ijkl})^2$ and $\|\rho\|^2 = \sum(\rho_{ij})^2$. Kim and Park(2002) obtained the asymptotic expansion of the mean distance up to order 3, that is,

$$
\begin{aligned}
& E[\gamma^2(X(t), m)] \\
= \;& nt - \frac{1}{6}\tau(m)t^2 + \frac{1}{90}\Big(- 6\Delta\tau(m) - \|R(m)\|^2 + \|\rho(m)\|^2\Big)t^3 + o(t^3) \quad \text{as} \quad t \downarrow 0.
\end{aligned}
$$

Let $p(t, x, y)$ be the fundamental solution of the following heat equation with

*This research was supported (in part) by KOSEF through Statistical Research Center for Complex Systems at Seoul National University.
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respect to the Riemannian volume $\sqrt{\det(g_{ij})(x)}dx$,

$$\begin{cases} \frac{\partial}{\partial t}u = \frac{1}{2}\Delta_M u \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where Δ_M is the Laplace-Beltrami operator for the Riemannian metric g on M, i.e,

$$u(t,x) = \int_M p(t,x,y)f(y)\sqrt{\det(g_{ij})(y)}dy$$

solves the above initial value problem. Watanabe (1987) introduced the notion of the generalized Wiener functional and the pull-back of Schwartz distribution under Wiener mappings in the framework of Malliavin calculus [see e.g. Malliavin (1978)]. By using these notions, he exploited the asymptotic expansion of the generalized Wiener functionals in the Sobolev space generated by *Ornstein-Uhlenbeck* (OU) operator and obtained the explicit expansion formula of the heat kernel.

In this paper we give the asymptotic expansion of the mean distance by using the expansion of of the fundamental solution of heat equation $p(t,x,y)$. We will use the summation convention, that is, we will omit the summation sign over repeated indics.

2 Preliminaries

Let $(\mathbb{W}_0^d, \mathbb{P})$ be the d-dimensional Wiener space and $\mathbb{H}$ be *the Cameron-Martin* subspace of $\mathbb{W}_0^d$. For a Hilbert space E, $\|\cdot\|_p$ denotes the $L^p(E)$-norm of E-valued Wiener functional. Let D be the Malliavin derivative and δ the adjoint operator of D which is a generalization of Ito integral and in this context is called *Skorohod integral.* Then $\mathcal{L} = \delta \circ D$ defines OU operator on $(\mathbb{W}_0^d, \mathbb{P})$. For $1 < p < \infty$ and $s \in \mathbb{R}$, we define a norm $\|\cdot\|_{p,s}$ by $\|F\|_{p,s} = \|(I - \mathcal{L})^{s/2}F\|_p$. The Banach space $\mathbb{D}_{p,s}(E)$ is the completion of the totality $P(E)$ of E-valued polynomials defined on $(\mathbb{W}_0^d, \mathbb{P})$ by the norm $\|\cdot\|_{p,s}$. Then we have

$$\mathbb{D}_{p',s'}(E) \hookrightarrow \mathbb{D}_{p,s}(E) \quad \text{for} \quad s \le s' \quad \text{and} \quad 1 < p \le p' < \infty,$$

where $\hookrightarrow$ denotes the continuous inclusion. Also we have

$$\mathbb{D}'_{p,s}(E) = \mathbb{D}_{q,-s}(E), \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad p \in (1,\infty),$$

where $(1/p) + (1/q) = 1$. Let us set

$$\mathbb{D}_\infty(E) = \bigcap_{1<p<\infty} \bigcap_{s>0} \mathbb{D}_{p,s}(E)$$

Then the space of generalized Wiener functional may be defined by

$$\mathbb{D}_{-\infty}(E) = \bigcup_{1<p<\infty} \bigcup_{s>0} \mathbb{D}_{p,-s}(E)$$

and

$$\tilde{\mathbb{D}}_{-\infty}(E) = \bigcap_{1<p<\infty} \bigcup_{s>0} \mathbb{D}_{p,-s}(E).$$

Let $F(\epsilon, w) \in \mathbb{D}_{\infty}(\mathbb{R}^d)$, $0 < \epsilon \leq 1$, and also $f_0(w), f_1(w), \cdots \in \mathbb{D}_{\infty}(\mathbb{R}^d)$. We say that the Wiener functional $F(\epsilon, w)$ has *the asymptotic expansion*

$$F(\epsilon, w) \sim f_0(w) + \epsilon f_1(w) + \cdots \quad \text{in} \quad \mathbb{D}_{\infty}(\mathbb{R}^d) \quad \text{as} \quad \epsilon \downarrow 0, \tag{2.1}$$

if for every $p \in (1, \infty)$, $s > 0$ and $n = 1, 2, \cdots$,

$$F(\epsilon, w) - [f_0(w) + \epsilon f_1(w) + \cdots + \epsilon^{n-1} f_{n-1}(w)] = O(\epsilon^n) \quad \text{in} \quad \mathbb{D}_{p,s}(\mathbb{R}^d) \quad \text{as} \quad \epsilon \downarrow 0. \tag{2.2}$$

We say that the family, $\{F(\epsilon, w) \in \mathbb{D}_{\infty}(\mathbb{R}^d) : \epsilon \in (0, 1]\}$, is said to be *uniformly nondegenerate* if for every $\epsilon \in (0, 1]$, $F(\epsilon, w)$ is nondegenerate and furthermore

$$\limsup_{\epsilon \to 0} \| [\det \sigma(\epsilon, w)]^{-1} \|_p < \infty \quad \text{for all} \quad p \in (1, \infty), \tag{2.3}$$

where $\sigma(\epsilon, w)$ is the Malliavin covariance of $F(\epsilon, w)$. The following Theorem is given by Watanabe (1987).

Theorem 1 *Let $F(\epsilon, w) \in \mathbb{D}_{\infty}(\mathbb{R}^d)$, $0 < \epsilon \leq 1$, be uniformly nondegenerate and has the asymptotic expansion (2.1). Then, for every $T \in \mathcal{S}'(\mathbb{R}^d)$, $T(F(\epsilon, w)) \in \tilde{\mathbb{D}}_{-\infty}$ has the asymptotic expansion in $\tilde{\mathbb{D}}_{-\infty}$:*

$$T(F(\epsilon, w)) \sim \Phi_0(w) + \epsilon \Phi_1(w) + \cdots \quad \textit{in} \quad \tilde{\mathbb{D}}_{-\infty}(\mathbb{R}^d) \quad \textit{as} \quad \epsilon \downarrow 0, \tag{2.4}$$

and $\Phi_0(w), \Phi_1(w), \cdots \in \tilde{\mathbb{D}}_{-\infty}(\mathbb{R}^d)$ are given by the formal Taylor expansion at f_0

$$T(f_0 + [\epsilon f_1 + \epsilon^2 f_2 + \cdots]) = \sum_{\alpha} \frac{1}{\alpha!} D^{\alpha} T(f_0) [\epsilon f_1 + \epsilon^2 f_2 + \cdots]^{\alpha} \tag{2.5}$$

$$= \Phi_0(w) + \epsilon \Phi_1(w) + \cdots, \tag{2.6}$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$ is a multi-index, $\alpha! = \Pi_{i=1}^{d} \alpha_i!$, $a^{\alpha} = \Pi_{i=1}^{d} a_i^{\alpha_i}$ for $a \in \mathbb{R}^d$ and

$$D^{\alpha} = \left(\frac{\partial}{\partial x^1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^d} \right)^{\alpha_d}.$$

In particular,

$$\Phi_0(w) = T(f_0(w)), \quad \Phi_1(w) = \frac{\partial}{\partial x^i} T(f_0(w)) f_1^i(w),$$

$$\begin{aligned} \Phi_2(w) &= \frac{\partial}{\partial x^i} T(f_0(w)) f_2^i(w) + \frac{1}{2!}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} T(f_0(w)) f_1^i(w) f_1^j(w), \\ \Phi_3(w) &= \frac{\partial}{\partial x^i} T(f_0(w)) f_3^i(w) + \frac{2}{2!}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} T(f_0(w)) f_1^i(w) f_2^j(w), \\ &\quad + \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} T(f_0(w)) f_1^i(w) f_1^j(w) f_1^k(w), \cdots. \end{aligned} \tag{2.7}$$

Here $D^\alpha T$ is the distributional partial derivative.

3 Asymptotic expansion of Wiener functional and Main Result

Now we derive the asymptotic expansion of $p(t, x, y)$, $x \neq y$, in powers of small time t. By a localization result [see e.g. Watanabe (1987)], we can reduce our problems to that of sde on $\mathbb{R}^d$. We take a normal coordinate system at $p \in M$ and extend this coordinate to the global Euclidean coordinate of $\mathbb{R}^d$. Hence in order to obtain the asymptotic expansion of $p(t, x, y)$ in small time, it is enough to consider the solution the sde on $\mathbb{R}^d$ over an d-dimensional Wiener space $(\mathbb{W}_0^d, \mathbb{P})$. This $p(t, 0, x)$ can be obtained by a generalized Wiener functional expectation

$$p(\epsilon^2, 0, x) = \mathbb{E}[\delta_x(X^\epsilon(1, 0, w))],$$

where $X^\epsilon(t, 0, w)$ is the solution of the following sde

$$\begin{cases} X^{\epsilon,k}(1) = \epsilon \int_0^1 \sigma_i^k(X^\epsilon(t)) dw_t^i + \epsilon^2 \int_0^1 \sigma_0^k(X^\epsilon(t)) dt \\ X^\epsilon(0) = 0. \end{cases} \tag{3.1}$$

By Cartan's formula, we have the following expansions:

Lemma 1 *For small x we have the expansion:*

$$\begin{aligned} \sigma_i^k(x) &= \delta_i^k + \frac{1}{6} R_{aibk} x^a x^b + \frac{1}{12} \nabla_a R_{bick} x^a x^b x^c + O(\|x\|^4). \\ \sigma_0^k(x) &= -\frac{1}{3}\rho_{ak} x^a + \frac{1}{24}(\nabla_k \rho_{ab} - 6\nabla_a \rho_{bk}) x^a x^b + O(\|x\|^3). \end{aligned}$$

Here the values of monomials in components of the curvature and its derivatives are taken at the origin.

Using Lemma 1, we have the expansion of $X^\epsilon(1,0,w)$.

Lemma 2 *For each ϵ, $0<\epsilon\le 1$, $X^\epsilon(1,0,w)\in\mathbb{D}_\infty(\mathbb{R}^d)$ has the asymptotic expansion*

$$X^\epsilon(1,0,w)\sim f_0(w)+\epsilon f_1(w)+\epsilon^2 f_2(w)+\cdots \quad \textit{in} \quad D_\infty(\mathbb{R}^d) \quad \textit{as} \quad \epsilon\downarrow 0$$

and $f_n(w)\in\mathbb{D}_\infty(\mathbb{R}^d)$, $n=0,1,\cdots$, are given as follows: for $k=1,\cdots d$,

$$f_0^k(w)=0,\quad f_1^k(w)=w^k(1),\quad f_2^k(w)=0,$$

$$f_3^k(w) \;=\; \frac{1}{6}R_{aibk}\int_0^1 w^a(t)w^b(t)dw_l^i-\frac{1}{3}\rho_{ak}\int_0^1 w^a(t)dt,\cdots$$

Let $F(\epsilon,w)=X^\epsilon(1,0,w)/\epsilon$. Then it is obvious that $F(\epsilon,w)$ is uniformly nondegenerate. Let $F_*(\mathbb{P})$ be probability law of F. Then $F_*(\mathbb{P})$ has the smooth density $p(\epsilon^2,0,x)$ and can be given by

$$p(\epsilon^2,0,x)=\mathbb{E}[\delta_x(X^\epsilon(1,0,w))]=\epsilon^{-d}\mathbb{E}[\delta_{x/\epsilon}(F(\epsilon,w))]$$

as generalized expectation of $\delta_{x/\epsilon}(F(\epsilon,w))$. Let $y=x/\epsilon$. By Theorem 1 and Lemma 2, $\delta_y(F(\epsilon,w))$ has an asymptotic expansion:

$$\begin{aligned}
\delta_y(F(\epsilon,w)) &= \delta_y(w(1)+[\epsilon^2 f_3(w)+\epsilon^3 f_4(w)+\epsilon^4 f_5(w)+\cdots]) \\
&= \delta_y(w(1))+\frac{\partial}{\partial x^i}\delta_y(w(1))[\epsilon^2 f_3^i(w)+\epsilon^3 f_4^i(w)+\epsilon^4 f_5^i(w)+\cdots] \\
&\quad +\frac{1}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\delta_y(w(1))[\epsilon^2 f_3^i(w)+\epsilon^3 f_4^i(w)+\epsilon^4 f_5^i(w)+\cdots] \\
&\quad \times[\epsilon^2 f_3^j(w)+\epsilon^3 f_4^j(w)+\epsilon^4 f_5^j(w)+\cdots]+\cdots \quad \text{in} \quad \tilde{\mathbb{D}}^{-\infty} \quad \text{as} \quad \epsilon\downarrow 0
\end{aligned} \tag{3.2}$$

From (3.2), $\mathbb{E}[\delta_y(F(\epsilon,w))]$ has the asymptotic expansion. From this we have that as $\epsilon\downarrow 0$,

$$\epsilon^d p(\epsilon^2,0,x) \;=\; \mathbb{E}[\delta_y(w(1))]+\epsilon^2\mathbb{E}[\frac{\partial}{\partial y^i}\delta_y(w(1))f_3^i(w)]+\cdots. \tag{3.3}$$

In order to compute generalized expectations in (3.3), we note that for $g\in\mathbb{D}_\infty(\mathbb{R}^d)$

$$\mathbb{E}[\delta_y(w(1))g(w)]=(-1)^{|\alpha|}D_y^\alpha\{\mathbb{E}[g(w)|w(1)=y](2\pi)^{-d/2}\exp[-\frac{|y|^2}{2}]\}|_{y=x/\epsilon}, \tag{3.4}$$

where D_y^α is the differentation with respect to y. It follows from (3.4) that we have

$$\mathbb{E}[\delta_y(w(1))] \;=\; (2\pi)^{-d/2}\exp[-\frac{|x|^2}{2\epsilon^2}]$$

$$\mathbb{E}[\frac{\partial}{\partial y^k}\delta_{x/\epsilon}(w(1))f_3^k(w)] = (\frac{1}{12}\tau + \frac{1}{18\epsilon^4}R_{aibk}x^a x^b x^i x^k - \frac{1}{12\epsilon^2}\rho_{ak}x^a x^k) \times (2\pi)^{-d/2}\exp[-\frac{|x|^2}{2\epsilon^2}].$$

Now we compute the expansion of mean distance of Brownian motion on M. Write $x^i = r\xi^i$ for $i = 1, 2, \cdots, d$ where $\xi \in \mathbb{S}^{d-1}$. First note that

$$R_{ijkl}\int_{\mathbb{S}^{d-1}} \xi^i\xi^j\xi^k\xi^l d\mathbb{S}^{d-1} = 0 \quad \text{and} \quad \rho_{ij}\int_{\mathbb{S}^{d-1}} \xi^i\xi^j d\mathbb{S}^{d-1} = \frac{1}{d}\omega_{d-1}\tau,$$

where ω_{d-1} is the area of the sphere $\mathbb{S}^{d-1}$. Using the spherical coordinates, we have

$$\begin{aligned}
&\mathbb{E}[d^2(0, X^\epsilon(1,0,w))] \\
= &\int_0^\infty \int_{\mathbb{S}^{d-1}} r^2(1 + \epsilon^2[\frac{1}{12}\tau + \frac{r^4}{18\epsilon^4}R_{ijkl}\xi^i\xi^j\xi^k\xi^l - \frac{r^2}{12\epsilon^2}\rho_{ij}\xi^i\xi^j] + \cdots) \\
&\times(2\pi\epsilon^2)^{-d/2}\exp[-\frac{r^2}{2\epsilon^2}]r^{d-1}dr d\mathbb{S}^{d-1} \\
= &\; d\epsilon^2 - \frac{1}{6}\tau(p)\epsilon^4 + o(\epsilon^6).
\end{aligned}$$

References

1. Ikeda, N and Watanabe, S. (1989), *Stochastic Differential Equations and Diffusion Processes*, Kodansha, Tokyo/North-Holland, Amsterdam.

2. Kim, Y.T. and Park, H.S. (2002), Mean distance of Brownian motion on a Riemannian manifold, *Stoch. Proc. and their Appl.*, 102, 117-138.

3. Malliavin, P. (1978), Stochastic calculus of variation and hypoelliptic operators, *Proc. Intern. Symp. SDE Kyoto 1976* (ed. by Ito), 195-263, Kinokuniya, Tokyo.

4. Malliavin, P. (1978), C^k-hypoellipticity with degeneracy, *Stochastic Analysis* (ed. A. Friedman and M. Pinsky), 199-214, 327-340, Academic Press, New York.

5. Watanabe, S. (1987), Analysis of Wiener functionals (Malliavin Calculus) and its Applications to Heat Kernels, *Ann. Probab.*, 15(1), 1-39.