Lie Algebraic Solution of Stochastic Differential Equations *

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We prove that the logarithm of the flow of stochastic differential equations is an element of the free Lie algebra generated by a finite set consisting of vector fields being coefficients of equations. As an application, we directly obtain a formula of the solution of stochastic differential equations given by Castell (1993) without appealing to an expansion for ordinary differential equations given by Strichartz (1987).

Key words and Phrases. Lie algebra, Specht-Wever theorem, iterated Stratonovich integral, Lie bracket, Stochastic differential equation.

1 Introduction

Suppose that $u(t)$ takes values in a C^∞ manifold and satisfies the initial value problem:

$$\begin{cases} u'(t) = A(t)(u(t)) \\ u(0) = a, \end{cases} \tag{1.1}$$

where $A(t)$ is a vector field on the manifold depending continuously on t. Then Strichartz (1987) has shown that the solution to the equation (1.1) is given by $u(t) = \exp(z(t))(a)$ and $z(t)$ is the generalized Campbell-Baker-Hasudroff-Dynkin formula:

$$z(t) \sim \sum_{m=1}^{\infty} \sum_{\sigma\in\sigma_m} \frac{(-1)^{e(\sigma)}}{m^2 \begin{pmatrix} m-1 \\ e(\sigma) \end{pmatrix}} \int \cdots \int_{0<s_1<\cdots<s_m<t}$$

$$\times [\cdots[A(s_{\sigma(1)}), A(s_{\sigma(2)})], \cdots], A(s_{\sigma(m)})]\cdots]ds_1\cdots ds_m, \text{ as } t \downarrow 0 \tag{1.2}$$

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where σ_m is the set of permutations on $\{1,\cdots m\}$, $e(\sigma)$ is the number of errors in ordering $\sigma(1),\cdots\sigma(m)$.

In this paper we consider an SDE (Stochastic Differential Equation) on $\mathbb{R}^d$,

$$\begin{cases} d\xi_t = \sum\limits_{i=1}^{r} X_i(\xi_t)\circ dB_t^i + X_0(\xi_t)dt \\ \xi_0 = x_0, \end{cases} \tag{1.3}$$

where $X_0,\cdots,X_r$ are C^∞ bounded vector fields on $\mathbb{R}^d$, $B_t = (B_t^1,\cdots,B_t^r)$ is an r-dimensional Brownian motion and the symbol $\circ d$ denotes the Stratonovich stochastic differential. In the past years, under some conditions on vector fields $X_0,\cdots,X_r$, a number of authors has expressed the solution as a functional of B_t. Recently, Castell (1993) has computed the asymptotic expansion in small time of the flow of SDE (1.3). For this he scales SDE (1.3) by introducing a small parameter $\epsilon > 0$:

$$\begin{cases} d\xi_t^\epsilon = \sum\limits_{i=1}^{r} \epsilon X_i(\xi_t^\epsilon)\circ dB_t^i + \epsilon^2 X_0(\xi_t^\epsilon)dt \\ \xi_0^\epsilon = x_0. \end{cases} \tag{1.4}$$

His method to get the expansion is essentially based on the well-known facts on a system of ordinary differential equations obtained by replacing the Brownian path in SDE (1.3) with the piecewise linear approximation. Applying (1.2) to the solution of this ordinary differential equation and then taking the limit for the extension to the case of SDE, he has proved the following Theorem.

Theorem 1 *For fixed $T > 0$, given $t < T$,*

$$\xi_t^\epsilon = \exp\left(\sum_{m=1}^{p-1} \epsilon^m \sum_{||\mathbf{J}||=m} C_t^{\mathbf{J}} X^{\mathbf{J}}\right)(x_0) + \epsilon^p R_p(\epsilon,t), \tag{1.5}$$

where the coefficients $C_t^{\mathbf{J}}$ of the Lie element $X^{\mathbf{J}}$ of the form

$$[X_{j_1}[X_{j_2}\cdots[X_{j_{m-1}}X_{j_m}]\cdots]$$

are linear combinations of the iterated Stratonovich integral. Here the remainder term $R_p(t,\epsilon)$ is bounded in probability as $\epsilon\to 0$, that is, there exist α and $c > 0$ such that for all $R > c$,

$$\lim_{\epsilon\to 0} P\left[\sup_{0\le s\le T} ||R_p(\epsilon,t)|| \ge R\right] \le \exp\left(-\frac{R^\alpha}{cT}\right).$$

In this paper, we consider the free algebra generated by $\mathfrak{X} = \{X_0, X_1,\cdots,X_r\}$ and extend this algebra to the algebra of formal power series related to the solution of SDE (1.3). An element of this algebra may be regraded as the solution of SDE (1.4) when this formal power series absolutely converges. We prove that $\log \xi_t^\epsilon$ is an element of the free Lie algebra generated by $\mathfrak{X} = \{X_0, X_1, \cdots, X_r\}$. As an application, we directly give the results of Castell (1993) with purely probabilistic and algebraic arguments, without appealing the expansions of the solution for an ordinary differential equation, given by Strichartz (1987).

2 Preliminaries

(1) Let $\mathbf{J} = (j_1, \cdots, j_m)$ be a multi-index with $m = 1, 2, \cdots$. We set $|\mathbf{J}| = m$ (size of $\mathbf{J}$) and $\|\mathbf{J}\| = m + \sharp\{\alpha | j_\alpha = 0\}$.

(2) Let $\mathfrak{X}(\mathbb{R}^d)$ be the totality of C^∞ bounded Lipschitz vector fields on $\mathbb{R}^d$ with the commutator $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{X}(\mathbb{R}^d)$. Let $X_i \in (\mathbb{R}^d)$, $i = 0, 1, \cdots, r$, be given. For $\mathbf{J} = (j_1, \cdots, j_m)$, we define $X^{\mathbf{J}}$ as follows:

$$X^{(j_1)} = X_{j_1} \quad \text{and} \quad X^{\mathbf{J}} = [X_{j_1}, [X_{j_2}, \cdots [X_{j_{m-1}}, X_{j_m}] \cdots].$$

Also we use the notation $X_{\mathbf{J}} = X_{j_1} \cdots X_{j_m}$.

(3) Let $(B_t^1, \cdots, B_t^r)$ be an r-dimensional Brownian motion. We write $B_t^{\mathbf{J}}$ as the iterated Stratonovich integral:

$$B_t^{\mathbf{J}} = \int_0^t \circ dB_{t_m}^{j_m} \int_0^{t_m} \circ dB_{t_{m-1}}^{j_{m-1}} \cdots \int_0^{t_2} \circ dB_{t_1}^{j_1}.$$

Also we write $B_t^0 = t$ for simplicity.

(4) Let σ_m be the permutation group on $\{1, \cdots m\}$. For each $\sigma \in \sigma_m$ we denote by $e(\sigma)$ the number of errors in ordering consecutive pairs $\sigma(i) < \sigma(i+1)$.

(5) If $\mathbf{J}$ is a multi-index of size m, then we denote $\mathbf{J} \circ \sigma = (j_{\sigma(1)}, \cdots, j_{\sigma(m)})$.

Now we consider the following formal power series in elements of the algebra $\overline{\mathfrak{F}}$ generated by $\mathfrak{X} = \{X_0, X_1, \cdots, X_r\}$:

$$\xi_t^\epsilon \;:=\; 1 + \sum_{p=1}^{\infty} \sum_{|\mathbf{J}|=p} \epsilon^{\|\mathbf{J}\|} B_t^{\mathbf{J}} X_{\mathbf{J}}, \tag{2.1}$$

where 1 denote the identity vector field.

Theorem 2 *Let $\xi_t(x_0)$ be the unique solution of SDE (1.4). Suppose that the infinite series (2.1) is absolutely convergent a.s. for each $t \in [0, T]$. Then it holds*

$$\xi_t^\epsilon(x_0) = x_0 + \sum_{p=1}^{\infty} \epsilon^p \sum_{\|\mathbf{J}\|=p} B_t^{\mathbf{J}} X_{\mathbf{J}}(x_0) \ \text{a.s.} \tag{2.2}$$

The proof of (2.2) is easily given by the successive applications of the Ito formula.

3 Main Results and Applications

We denote by $\mathfrak{F}_L$ Lie algebra of an associative algebra $\mathfrak{F}$. Let $\mathfrak{F}^{\mathcal{L}}$ be the subalgebra of $\mathfrak{F}_L$ generated by the elements of $\mathfrak{X}$. An element $a \in \mathfrak{F}$ is called a *Lie element* if $a \in \mathfrak{F}^{\mathcal{L}}$. We see that an element is a Lie algebra if and only if its homogeneous parts are Lie elements. Also if a is a Lie element which is homogeneous of degree m, then a may be written as a linear combination of elements of the following form:

$$[\cdots[[X_{i_1}, X_{i_2}], X_{i_3}], \cdots X_{i_m}], \quad \text{for} \quad i_j = 0, 1, \cdots, r \quad \text{and} \quad m = 1, 2, \cdots.$$

We set $\xi_t^\epsilon := \exp Z^\epsilon(t)$, where

$$Z^\epsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{l=1}^{\infty} \epsilon^l \sum_{\|\mathbf{J}\|=l} X_{\mathbf{J}} B_t^{\mathbf{J}} \right)^n. \tag{3.1}$$

We apply the criterion of Friedrichs to show that the element $\log(\xi_t^\epsilon)$ is a Lie element, which is the key Lemma in this paper.

Theorem 3 *For fixed $t \geq 0$ and $0 \leq \epsilon \leq 1$, the element $\log(\xi_t^\epsilon)$ is a Lie element, that is, $Z^\epsilon(t) \in \mathfrak{F}_L$.*

Using Theorem 3, we directly solve SDE without appealing an associated ordinary differential equation. For fixed positive integers k we consider all ways of choosing positive integers $p_1, \cdots, p_n$ satisfying $p_1 + \cdots + p_n = k$. Let us set $q_0 = 0$ and $q_j = p_1 + \cdots + p_j$ for $j \geq 1$. We denote $\mathbf{J}_i = (j_{q_i+1}, j_{q_i+2}, \cdots, j_{q_{i+1}})$ for $i = 0, 1, \cdots, n-1$, and $X_{j_\alpha}^\epsilon = \epsilon^{\|j_\alpha\|} X_{j_\alpha}$, where the notation $\|j_\alpha\|$ is defined by analogy with $\|\mathbf{J}\|$,

$$\|j_\alpha\| = \begin{cases} 1 & \text{if } j_\alpha \neq 0 \\ 2 & \text{if } j_\alpha = 0. \end{cases}$$

Then $Z^\epsilon(t)$ can be written as

$$Z^\epsilon(t) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \sum_{\substack{p_1, \cdots, p_n \\ p_1 + \cdots + p_n = k}} \frac{(-1)^{n-1}}{n} \int \circ dB_{t_1}^{j_1} \circ dB_{t_2}^{j_2} \cdots \circ dB_{t_{q_n}}^{j_{q_n}} X_{j_1}^\epsilon X_{j_2}^\epsilon \cdots X_{j_{q_n}}^\epsilon, \tag{3.2}$$

where the integral in the second equation is taken over the following region:

$$0 < t_1 < \cdots < t_{q_1} < t, \cdots, 0 < t_{q_{n-1}+1} < \cdots < t_{q_n} < t.$$

Let $\mathfrak{F}'$ denote the ideal $\mathfrak{M} \oplus (\mathfrak{M} \otimes \mathfrak{M}) \oplus \cdots$ in $\mathfrak{F}$. Then we define a linear mapping τ of $\mathfrak{F}'$ into $\mathfrak{F}^{\mathcal{L}}$ such that

$$\tau X_i = X_i, \quad \tau(X_{j_1} \cdots X_{j_m}) = [\cdots [X_{j_1}, X_{j_2}] \cdots X_{j_m}], \quad \text{for} \quad m > 1.$$

Theorem 4 (Spechet-Wever) *If a field Φ is of characteristic 0, then a homogeneous element a of degree $m > 0$ is a Lie element if and only if $\tau a = ma$.*

Since $Z^\epsilon(t)$ is a Lie element, if we apply the operator τ to the homogeneous part of degree k we have the following expression of $Z^\epsilon(t)$ as a Lie element by Spechet-Wever Theorem:

$$\begin{aligned} Z^\epsilon(t) \;&=\; \sum_{k=1}^{\infty}\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1,\cdots,p_n \\ p_1+\cdots+p_n=k}} \int \circ dB_{t_1}^{j_1} \circ dB_{t_2}^{j_2} \cdots \circ dB_{t_{q_n}}^{j_{q_n}} \\ & [\cdots [X_{j_1}^\epsilon, X_{j_2}^\epsilon] \cdots X_{j_{q_n}}^\epsilon]. \end{aligned}$$

By using the algebraic computations in Strichartz (1987) , we obtain the following explicit representation:

$$Z^\epsilon(t) = \sum_{m=1}^{\infty} \epsilon^m \sum_{\|\mathbf{J}\|=m} C_t^{\mathbf{J}} X^{\mathbf{J}}, \tag{3.3}$$

where

$$C_t^{\mathbf{J}} = \sum_{\sigma \in \sigma_{|\mathbf{J}|}} \frac{(-1)^{e(\sigma)}}{|J|^2 \begin{pmatrix} |J|-1 \\ e(\sigma) \end{pmatrix}} B_t^{\mathbf{J} \circ \sigma^{-1}}.$$

Now we consider the asymptotic expansion given by Theorem 1. Let $\xi_t^\epsilon(x_0)$ be the solution of SDE (1.4). For fixed $T > 0$, we define $H_p(\epsilon, t)$, $t \leq T$ and $0 \leq \epsilon \leq 1$, by

$$\xi_t^\epsilon(x_0) = 1 + \sum_{k=1}^{p-1} \epsilon^k \sum_{\|\mathbf{J}\|=k} X_{\mathbf{J}}(x_0) B_t^{\mathbf{J}} + \epsilon^p H_p(\epsilon, t) \quad \text{a.s.} \tag{3.4}$$

Then Azencott (1982) has shown that there exist $\alpha > 0$ and $\beta > 0$ such that for all $c > \beta$,

$$\lim_{\epsilon \to 0} P\left[\sup_{0 \leq s \leq T} \|H_p(\epsilon, t)(x_0)\| \geq c\right] \leq \exp\left(-\frac{c^\alpha}{\beta T}\right).$$

We recall the following definition, introduced by Azencott (1982) [see also Castell(1993)]:

Definition 1 *Let ζ be a stopping time and let X_t be a continuous stochastic process on $[0,\zeta)$ with values in $\mathbb{R}^p$. Then X is said to be in $\mathcal{W}(\alpha,c,\zeta)$ if and only if for all $t\geq 0$ and $R\geq c$,*

$$P\left[\sup_{0\leq s\leq t}\|X_s\|\geq R; t<\zeta\right]\leq \exp\left(-\frac{R^\alpha}{ct}\right).$$

If we define

$$Z^\epsilon_{p-1}(t)=\sum_{m=1}^{p-1}\epsilon^m\sum_{\|\mathbf{J}\|=m}C^{\mathbf{J}}_t X^{\mathbf{J}},$$

then

$$\begin{aligned}\exp(Z^\epsilon_{p-1}(t)) &= 1+\sum_{m=1}^{\infty}\epsilon^m\sum_{a=1}^{m}\frac{1}{a!}\sum_{\|\mathbf{J}_1\|+\cdots+\|\mathbf{J}_a\|=m}C^{\mathbf{J}_1}_t X^{\mathbf{J}_1}\cdots C^{\mathbf{J}_a}_t X^{\mathbf{J}_a}\\ &\quad+\epsilon^p Q^{(1)}_p(t)+\epsilon^p Q^{(2)}_p(\epsilon,t).\end{aligned}$$

Here $Q^{(1)}_p(t)(x_0)\in\mathcal{W}(\alpha,c,T)$ for each $T>0$ and $\lim_{\epsilon\to 0}\sup_{0\leq t\leq T}\|Q^{(2)}_p(\epsilon,t)(x_0)\|=0$ (a.s.). It follows from (3.4) that

$$\xi^\epsilon_t(x_0)=\exp Z^\epsilon_{p-1}(t)(x_0)+\epsilon^p H_p(\epsilon,t)(x_0)-\epsilon^p Q^{(1)}_p(t)(x_0)+\epsilon^p Q^{(2)}_p(\epsilon,t)(x_0).$$

Hence we have Theorem 1 given by Castell (1993).

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