Natural Element Method를 이용한 전자장 해석

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Electro-Magnetic Field Computation Using the Natural Element Method

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Abstract - The natural element method is a kind of meshless Galerkin method. The shape function is derived from the natural neighbor coordinates interpolation scheme. Natural neighbor shape functions are C^1 everywhere, except the nodes where they are C^0. The numerical integration is carried out using the Delaunay triangles as the background cells. The method is applied to the test problems and simulation results show that the natural element method can give accurate solutions for the electromagnetic field problems..

1. Introduction

Recently, many meshless methods have been developed in solving the electromagnetic field problems [1-5]. These methods do not need mesh generation procedure and only data sets for node distribution and description of the boundaries are required. Most of the meshless methods such as the Element-Free Galerkin (EFG) method [3,4] employ the moving least square (MLS) approximation to compute the shape function. Because the shape functions of the EFG methoddo not have the Kronecker delta function property [6], special treatment for imposing the essential boundary conditions is required [7]. The natural element method (NEM) is a kind of meshless method employing the Galerkin scheme for the solution of partial differential equations. The difference from other meshless methods is that the trial functions are constructed using the natural neighbor coordinates. The shape function of the NEM has the Kronecker delta function property and the essential boundary conditions can be imposed directly like the finite element method [8,9].

In this paper, the description of the NEM and the application of the method to the electromagnetic field computations are presented. By some numerical experiments, the rate of convergence and the accuracy of the method are shown.

2. Voronoi Diagram and Delaunay Tessellation

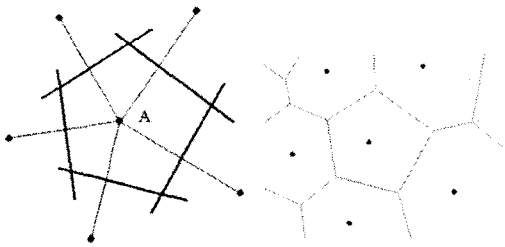
The Voronoi diagram and Delaunay tessellation are widely used in the field of computational geometry in constructing the geometries. For Euclidean space R2, consider a set of distinct points P = { p_1 , p_2, ... , p_N }. The Voronoi polygon of the point p_n is defined as follows:

$$T_n = \{ x \in \mathbf{R}^2 : d(x, x_n) < d(x, x_m) \ \forall m \neq n \} \qquad (1)$$

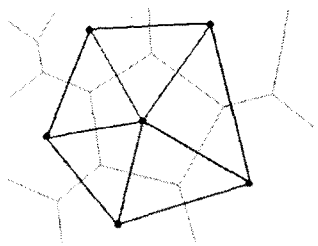
where $d(x_m, x_n)$ is the distance between x_m and x_n.

Each T_n is the intersection of finitely many open half-spaces, each being delimited by the perpendicular bisector. The Voronoi polygon can be viewed as the locus of all points closer to p_n than to any other nodes. The Voronoi diagram for a set of nodes divides the plane into a set of regions, one for each node, such that any point in a particular region is closer to that region's node than to any other node.

Fig. 1(a) shows the some nodes and the Voronoi cell for node A and Fig. 1(b) shows the Voronoi diagram for given nodes. The Delaunay triangulation is then constructed by connecting the nodes whose Voronoi cells have the common boundaries as shown in Fig. 1(c). Therefore Delaunay triangulation and Voronoi diagram are dual structures.



(a) Voronoi cell for node A (b) Voronoi diagram



(c) Delaunay triangulation

Fig. 1. Voronoi diagram and Delaunay triangulation

3. Natural Neighbor Interpolation

If T_n and T_m have a common boundary, p_n and p_m are considered as neighbors. The notion of a set of neighboring nodes is generalized by the definition of natural neighbor nodes. The natural neighbors of any node are those in the neighboring Voronoi cells, or equivalently, those to which the node is connected by the sides of the Delaunay triangle [8,9].

If the sampling point x in the Delaunay triangulation is given, the natural neighbors of x are the set of nodes which are connected to it. The number of natural neighbors is a function of position x and depends on the local nodal density. Consider an interpolation scheme for a function $u(x)$ in the form of (2).

$$u^h(x) = \sum_{I=1}^{n} \phi_I(x)\, u_I \tag{2}$$

where u_I (I = 1, 2, ... , n) is the function value at neighboring node I, and $\phi_I(x)$ is the weight associated with each node.

The weight $\phi_I(x)$ in the NEM is taken as the natural neighbor coordinates of the point x in the plane. Fig. 2 shows an example of the Voronoi cells. In Fig. 2(a), a point x is introduced into the Voronoi diagram and it is observed that x has four natural neighbors, namely nodes 1-4. If a point x is added, then a new Voronoi cell is placed around x as shown in Fig. 2(b).

The natural neighbor coordinates of x with respect to a natural neighbor I is defined as the ratio of the area of their overlapping Voronoi cells to the total area of the Voronoi cell for x [8].

$$\phi_I(x) = A_I(x)/A(x) \tag{3}$$

where I = 1, 2, ... , n , and $A(x) = \sum_{J=1}^{n} A_J(x)$.

For example, in Fig. 2, the four regions composing the closed polygon *abcd* are called the second-order cells and their union (polygon *abcd*) is a first-order Voronoi cell. The natural neighbor coordinates or shape function $\phi_1(x)$ for node 1 is given by (4).

$$\phi_1(x) = A_{abfe}/A_{abcd} \tag{4}$$

From the definition of $\phi_I(x)$, the shape function has following properties:

$$0 \le \phi_I(x) \le 1 \ , \ \phi_I(x_J) = \delta_{IJ} \ , \ \sum_{I=1}^{n} \phi_I(x) = 1 \tag{5}$$

The Kronecker-delta property of (5) is one of the main different properties compared with MLS approximation scheme. In one-dimension, the shape function by the natural neighbor interpolation is precisely the same as the 1-D linear finite element shape function [8]. Another important property of natural neighbor shape functions is they are C^0everywhere, except the nodes where they are C^0.

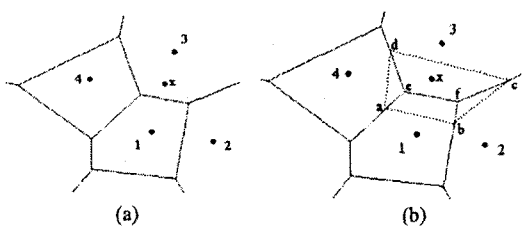


Fig. 2. Construction of natural neighbor coordinates : (a) Original Voronoi diagram , (b) 1st and 2nd Voronoi cells for point x

4. Implementation of the NEM

The implementation of the NEM is similar to that of FEM or other Galerkin procedure-based meshless methods. To construct the shape function and compute their derivatives, the number of natural neighbors and their global nodal numbers along with the number of triangles associated with the natural neighbors as well as their global triangle numbers should be computed. The simple means to determine if a node is a natural neighbor of a point x is to use the following criterion.

$$\| v - x \| < R \tag{6}$$

where v is the center of the circumcircle of a triangle and R is its radius.

If the condition (6) for a triangle whose nodes are n_I, n_J and n_K is satisfied, then the nodes n_I, n_J and n_K are the natural neighbors of the point x.

The system matrix and forcing vector by the NEM are assembled on a nodal basis. For the numerical integration, the integration cells are needed. In this paper, the Delaunay triangles themselves are used as the integration cells. The essential boundary conditions can be directly enforced like the finite element procedure because the shape function has Kronecker delta property and the trial function of NEM is strictly linear between two nodes that belong to an edge of a Delaunay triangle on the boundary of the convex hull [8]. Generally the system matrix from the NEM is symmetric and sparse but not necessarily banded.

5. Simulation and Results

First example is the 2-D simple test problem to validate the NEM. The equation to be solved and the exact solution are given by:

$$\Delta u = -8\pi^2 \sin(2\pi x)\ \sin(2\pi y) \tag{7}$$
$$u(x,y) = \sin(2\pi x)\ \sin(2\pi y) \tag{8}$$

In performing the numerical integration, the three-point Gauss quadrature is used for each Delaunay triangle. The numerical solution of $u(x,y)$ by the NEM is plotted in Fig. 3 when the node distribution of 21 × 21 grid with regular node-spacing is used. In this case, the maximum relative error of u compared with the exact solution for each node is less than 1[%]. Fig. 4 shows the relative L2 (Ω) error norm with respect to the nodal spacing for regular grid. The L2 (Ω) error norm is defined as follows:

$$\left\| \mathbf{u} - \mathbf{u}^h \right\|_{L^2(\Omega)} = \left(\int_\Omega (u_i - u_i^h)(u_i - u_i^h) d\Omega \right)^{1/2} \tag{9}$$

where $\mathbf{u}$ and $\mathbf{u}^h$ are the exact and numerical solutions, respectively.

As shown in the figure, it is found that the convergence rate of the NEM for this problem is better than that of the FEM.

The NEM is also applied to the analysis of electrostatic field. The problem definition is shown in Fig. 5(a) and nodal discretization is shown in Fig. 5(b). Fig. 6 shows the comparison of the NEM results with FEM ones. As shown in the figure, the NEM results show very good agreement with FEM ones. However, the computational cost of the NEM is somewhat higher because the NEM requires the more steps for Delaunay tessellation, searching routine for natural neighbors and so on.

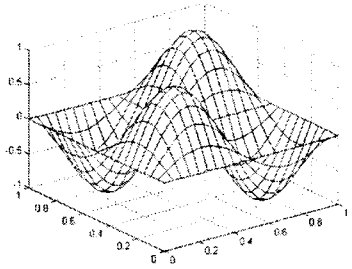


Fig. 3. Plot of numerical solution

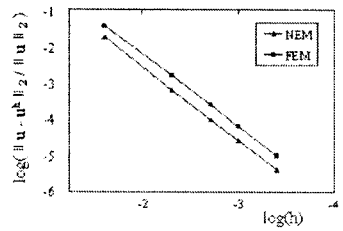
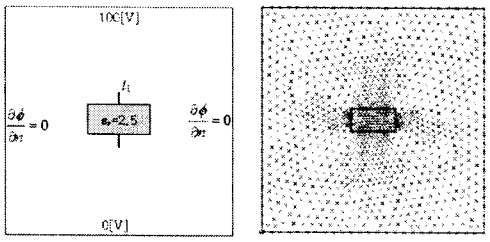
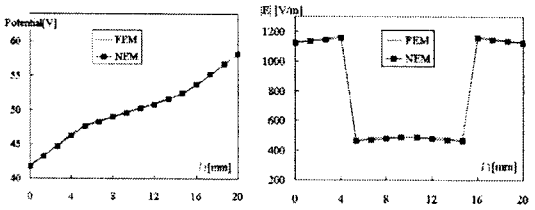


Fig. 4. Relative error norm (h is nodal spacing).



(a) (b)

Fig. 5. Application of the NEM to the electrostatic problem : (a) Problem definition , (b) Nodal discretization



(a) Potential distribution (b) Electric field intensity

Fig. 6. Comparison of NEM results with FEM ones along line *l*1

6. Conclusions

In this paper, the description of the NEM and the application of the method to the electromagnetic field computation are presented. The shape function of the NEM is derived from the natural neighbor coordinates and has many common properties in comparison with the FEM. It is found that the NEM is a promising numerical method for electromagnetic field computations and can be employed in many applications such as shape optimization, 3-D problem, moving boundary problem and so on.

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7. References

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