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불연속 리아푸노프 함수를 이용한 입력제한이 있는 불확실 선형 시스템의 안정성 해석

Robust stability analysis of uncertain linear systems with input saturation using piecewise Lyapunov functions

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Abstract : In this paper, we consider the problem of finding the stability region in state space for uncertain linear systems with input saturation. For stability analysis, two Lyapunov functions are chosen. One is for the linear region and the other is for the saturated region. Piecewise Lyapunov functions are obtained by solving successive linear matrix inequalites(LMIs) relaxations. A sufficient condition for robust stability is derived in the form of stability region of initial conditions. A numerical example shows the effectiveness of the proposed method.

Keywords : Saturation, Piecewise Laypunov Functions, LMIs

Ⅰ. Introduction

Saturation nonlinearities are shown in virtually all physical systems even though the degree of importance may vary. Input saturation often causes the stability problem or aggravation of the performance of the control systems.

Therefore, the stability analysis for the linear system subject to input saturation has been studied by many researchers in past decades [1,2,3,4]. In general, global asymptotic stability of the linear system with input saturation cannot be achieved by linear feedback control laws. In recent works, the notion of semi-global stability of linear system with input saturation was introduced by Sussman and Yang (1991)[6].

Henrion and Tarbourich (1999)[2] considered the problem of finding the stability region for an uncertain linear system with input saturation using LMI based computational formulas. They formulated the saturation nonlinearity using a saturation index and modelled the actuator saturation as polytopic system uncertainties under given state feedback. They changed the stability problem of the system into a BMI problem and relaxed the BMI problem to dual LMI problems. So they found the greatest set of initial condition under their sufficient condition. In this paper, we propose a new method to develop a less conservative sufficient condition for the stability of uncertain linear system with input saturation. It is based on piecewise quadratic Lyapunov functions.

We describe the state feedback control system with saturation as a switching system where switching occurs between linear region and saturated region. Then we choose two different Lyapunov functions at distinct convex areas according to the given saturation index and uncertainties.

We use a dual LMI relaxation method to find the greatest set of stability region under our sufficient condition. Finally, we demonstrate the effectiveness of the proposed approach using a numerical example.

Ⅱ. Problem statement and preliminaries

Consider the uncertain linear system with input saturation given by:

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)sat(u(t)), \quad (1)$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input vector.

We choose the state feedback control law as

$$u(t) = Kx(t). \quad (2)$$

The component of input vector $u(t)$ is formulated

$$u_i = sat[K_i x] = \begin{cases} w_i & \text{if } K_i x > w_i \\ K_i x & \text{if } |Kix| \le w_i \\ -w_i & \text{if } K_i x < -w_i \end{cases} \quad (3)$$

where w_i is a given positive scalar and K_i is the ith row of matrix K.

The parameter uncertainties considered here are assumed to be norm bounded and of the form

$$[\Delta A \quad \Delta B] = DF(t)[E_1 \; E_2], \quad (4)$$

where D, E_1, and E_2 are known constant real matrices of appropriate dimensions and $F(t) \in R^{d \times e}$ is an unknown matrix function satisfies $F^T(t)F(t) \le I$, (5)

Then the system (1) can be rewritten as

$$\dot{x}(t) = \overline{A}x(t) + \overline{B}GFx(t), \quad (6)$$

where $\overline{A} = A + \Delta A$, $\overline{B} = B + \Delta B$, G is a diagonal matrix and whose diagonal element is

$$g_i(x) = \begin{cases} w_i/K_i x & \text{if } K_i x > w_i \\ 1 & \text{if } |K_i x| \le w_i \\ -w_i/K_i x & \text{if } K_i x < -w_i \end{cases} \quad (7)$$

In order to formulate the linear region and saturated region, define

$$S(F, w) = \{x \in R^n; -w_i \le K_i x \le w_i, i = 1, \ldots, m\} \quad (8)$$

and a given positive scalar g_i, we define the symmetric polyhedron

$$S_g(F, w) = \{x \in R^n; -w_i/g_i \le K_i x \le w_i/g_i, \; i = 1, \ldots, m\}, \quad (9)$$

a contractively invariant set

$$E(P, \rho) = \{x \in R^n : x^T P x \le \rho\}, \quad (10)$$

where $P \in R^{n \times n}$ is a positive definite matrix.

The following lemma will be used to find sufficient condition for the stability of the system (6).

Lemma 1.[4] Given an ellipsoid $E(P, \rho)$, if there exists an $H \in R^{m \times n}$ such that

$$(A + BM(v, F, H))^T P + P(A + BM(v, F, H)) < 0 \quad (11)$$

where $M(v, F, H) = \begin{bmatrix} v_1 f_1 + (1 - v_1 h_1) \\ \cdots \\ v_m f_m + (1 - v_m h_m) \end{bmatrix}$

for all $v \in V$ and $E(P, \rho) \subset L(H)$, i.e., $|h_i x| \le 1$ for all $x \in E(P, \rho), i \in [1, m]$, then $E(P, \rho)$ is a contractively invariant set.

III. Stability analysis using Piecewise Lyapunov Functions

In this section, we investigate the stability of uncertain linear system with input saturation. We change the stability problem of the uncertain linear system with input saturation into that of linear switching systems. Figure 1 describes the switching mechanism of the system whose saturation is switched between a linear region and a saturated region. In figure 1, the single input system is illustrated for simplicity. Therefore, the system (6) should be expressed as a switching system with m switching.

For the stability of switching linear systems, the stability for all switching sequences among asymptotically stable A_i-matrices should be guaranteed. The sufficient condition is derived from the existence of $P = P^T > 0$ satisfying $A_i^T P + P A_i < 0$, for i=1,...m.

In this paper, we will propose piecewise Lyapunov functions for the stability of the uncertain linear system with input saturation.

Let Ω_q denote a region where one searches for a P_q in the quadratic Lyapunov-like functions $V_q = x^T P_q x$ that satisfies the stability condition

$$\dot{V}_q(x) = [\frac{\partial}{\partial x} V_q(x)] A_i x = x^T (A_i^T P_q + P_q A_i^T) x \le 0. \quad (12)$$

Additionally, the LMI problem formulation requires switching condition which is uses Lyapunov-like functions V_r, then

$$V_r(x) < V_q(x). \quad (13)$$

By using the so-called S-procedure, we provide the following theorem.

Theorem 1. Given a matrix G, The system (1) is locally quadratically stable in Lyapunov level set $E(P_1, \rho) \cup E(P_2, \rho)$ if there exist an $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$ and $\lambda_1 > 0, \lambda_2 > 0$ such that

$$(\overline{A} + \overline{B}F)^T P_1 + P_1(\overline{A} + \overline{B}F) + \lambda_1(P_2 - P_1) < 0, \quad E(P_1, \rho) \subset L(F)$$

$$(\overline{A} + \overline{B}GF)^T P_2 + P_2(\overline{A} + \overline{B}GF) + \lambda_2(P_1 - P_2) < 0,$$

$$E(P_2,\rho) \subset L(GF) \quad (14)$$

Proof. Consider the following Lyapunov function candidate

$$V(t)=\begin{cases} x^TP_1x & \text{if } x \in S(F,w) \\ x^TP_2x & \text{if } x \in S_g(F,w) \end{cases}. \quad (15)$$

For a initial state $x(0)$, the state is switched from a saturated region to linear region and vice versa. To verify the stability of the system (1), we only need to check that

$$dV(x(t))/dt < 0. \quad (16)$$

Note that the derivative is given by

$$dV(x(t))/dt=\begin{cases} x'(\overline{A}+\overline{B}F)^TP_1+P_1(\overline{A}+\overline{B}F))x \\ \qquad when\ x^TP_2x \geq x^TP_1x, \\ x'(\overline{A}+\overline{B}GF)^TP_2+P_2(\overline{A}+\overline{B}GF))x \\ \qquad when\ x^TP_1x \geq x^TP_2x. \end{cases} \quad (17)$$

From (17), the condition for robust stability of (1) is obtained by using an S-procedure. ■

At this point, we apply the lemma 1 to the piecewise Lyapunov function, then the condition can be stated as follows.

Theorem 2. Given an ellipsoid $E_F(P_1,\rho)$ and $E_H(P_2,\rho)$, if there exist an λ_1,λ_2 and $H \in R^{m\times n}$ such that

$$(\overline{A}+\overline{B}GF)^TP_1+P_1(\overline{A}+\overline{B}GF)+\lambda_1(P_2-P_1)<0, \quad E(P_1,\rho)\subset L(GF)$$

$$(\overline{A}+\overline{B}H)^TP_2+P_2(\overline{A}+\overline{B}H)+\lambda_2(P_1-P_2)<0, \quad E(P_2,\rho)\subset L(H) \quad (18)$$

and $|H_ix| \leq w_i$

then $E_F(P_1,\rho)\cup E_H(P_2,\rho)$ is a contractive invariant set and hence inside the domain of attraction.

Proof: The theorem 1 is easily extended by introducing the saturation level matrix G and an auxiliary matrix H.

For the region $S_g(F,w)$, let

$$V_1(t)=x^TP_1x. \quad (19)$$

We need to show that

$$\dot{V}_1(x)=2x^TP_1(\overline{A}+\overline{B}GFx)<0 \quad x\in E_{F(P_1,\rho)}\{0\}. \quad (20)$$

Here we have

$$\dot{V}_1(x)=2x^T\overline{A}^TP_1x+\sum_{i=1}^{m}2x^TP_1\overline{b}_ig_if_ix<0. \quad (21)$$

For each term $2x^TP_1\overline{b}_ig_if_ix$,

If $|g_if_ix| \geq w_i$, then

$$2x^TP_1\overline{b}_ig_if_ix \leq 2x^TP_1\overline{b}_ih_ix$$

where $|h_ix| \leq w_i$, for *all* $x \in E(P_1,\rho)$.

For the region $S_H(F,w)=\{x| \ |h_ix| \leq w_i\}$, define

$$V_2(t)=x^TP_2x. \quad (22)$$

It follows that

$$\dot{V}_2(x)=2x^T(\overline{A}^T+\overline{B}H)^TP_2x<0, \text{ for } all x \in E_H(P_2,\rho). \quad (23)$$

For switching, the stability of the system is guaranteed using a S-procedure. ■

With all the level sets satisfying the condition of Theorem 2, we may choose the "largest" one to obtain the less conservative region of attraction.

In [3], a sufficient condition for the system (1) to be robustly stable is introduced using a common quadratic Lyapunov function with the following LMI.

$$\begin{bmatrix} (A+BGK)^TP+P(A+BGK) & * & * \\ D^TP & -I & * \\ E_1+E_2GK & 0 & -I \end{bmatrix}<0. \quad (24)$$

The constraint $E(P_2,\rho)\subset L(H)$ and $|H_ix|=w_i$ is equivalent to

$$\min\{x^TP_2x: h_{ix}=w_i\} \geq \rho, \ i=1,2,\ldots,m. \quad (25)$$

By using the Lagrange multiplier, we obtain the following equivalent condition

$$\rho h_iP^{-1}h_i^T \leq 1, \ i=1,2,\ldots,m. \quad (26)$$

By pre- and post-multiplying $Q_i=P_i^{-1}$, $i=1,2$ on the both sides of (18) and using the well known inequality

$$X^TY+Y^TX \leq \epsilon X^TX+\frac{1}{\epsilon}Y^TY, \text{ for } any\ \epsilon>0. \quad (27)$$

We obtain the following theorem from theorem 2 using the above matrix inequality.

Theorem 3. Given an $\lambda_i>0, \rho>0, G_\gamma$, uncertain saturated system (1) is locally quadratically stable in the Lyapunov level set $E_F(P_1,\rho)\cup E_H(P_2,\rho)$ if there exist an

$\epsilon > 0$, $Z \in R^{m \times n}$, $Q_1 > 0$ and $Q_2 > 0$ such that

$$\begin{vmatrix} \Phi_1 & * & * & * \\ \epsilon D^T & -\epsilon I & * & * \\ (E_1 + E_2 K) Q_1 & 0 & -\epsilon I & * \\ Q_1 & 0 & 0 & -\frac{1}{\lambda_1} Q_2 \end{vmatrix} < 0, \quad (28)$$

$$\begin{vmatrix} \Phi_2 & * & * & * \\ \epsilon D^T & -\epsilon I & * & * \\ E_1 Q_2 + E_2 Z & 0 & -\epsilon I & * \\ Q_2 & 0 & 0 & -\frac{1}{\lambda_2} Q_1 \end{vmatrix} < 0, \quad (29)$$

$$\begin{vmatrix} \rho w_i^2 & \gamma_i K_i Q_1 \\ \gamma_i Q_1 K_i^T & Q_1 \end{vmatrix} \geq 0, \quad \begin{vmatrix} \rho w_i^2 & \gamma_i Z_i \\ \gamma_i Z_i^T & Q_2 \end{vmatrix} \geq 0, \quad (30,31)$$

$$\Phi_1 = Q_1 (A + BK)^T + (A + BK) Q_1 + \lambda_1 Q_1,$$

$$\Phi_2 = Q_2 A^T + Z^T B^T + A Q_2 + B Z + \lambda_2 Q_2.$$

Note that the matrix inequalities (28)-(31) is BMIs. Since the BMIs become linear when one decision variable is fixed, we used it for schemes based on LMI relaxations[3]. The algorithm will solve the BMIs problems as follows,

Step 1. Let $G_\gamma = I, \rho = 1, \epsilon = 1$.

Solve for Q_1, Q_2, Z the LMI set (28)-(31).

Step 2. Given Q_1, Q_2, Z, solve for G_γ, ρ, ϵ,

$\min (tr(G_\gamma) + \rho)$

Step 3. Given G_γ, ρ, ϵ.

Solve for Q_1, Q_2, Z, $\min (tr(Q_2))$

Step 4. If some conditions on the size of E are fulfilled, then stop, otherwise, go to Step 2.

Ⅳ. Numerical Example

Consider the system with state feedback $u(t) = Kx(t)$ [3],

$$\dot{x}(t) = (A + DF(t)E_1)x(t) + (B + DF(t)E_2) sat(u(t)), \quad (32)$$

where

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \; D = \frac{1}{20} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$E_1 = E_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; K = -\begin{bmatrix} 0.7283 & 0.0338 \\ 0.0135 & 1.3583 \end{bmatrix},$$

$w_1 = 5, w_2 = 2$.

Graphical comparisons between our regions of attraction and that obtained by the method in Henrion and Tarbouriech (1999) are shown in Fig 2.

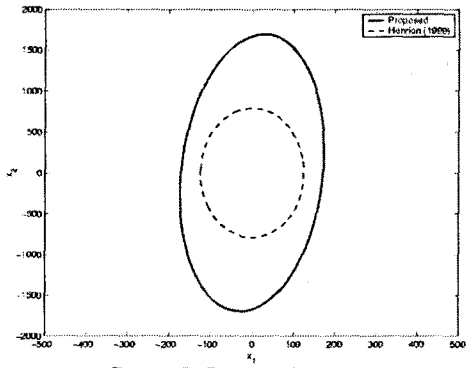

Figure 2: Region of attraction

Ⅴ. Conclusions

In this paper, a new technology is presented to obtain a less conservative sufficient condition for the stability of the uncertain linear system with input saturation. Piecewise Lyapunov functions are determined to find the region of attraction of the system. An LMI relaxation method is used to solve the BMI problem. Numerical example demonstrates the effectiveness of this method.

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