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연속 압축재의 유효길이 계수

Effective Length Factors for Continuous Compression Members

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국문초록

보 유사법을 이용하여 연속 압축재의 유효길이 계수를 각 경간별로 결정하는 방법을 제안한다. 제안하는 보 유사법은 4가지 단계로 요약할수 있는데 그 첫 단계는 주어진 압축재를 이것과 동일한 단면성능 및 경간을 갖는 연속보로 치환하는 것이다. 제 2단계에서는 연속보 각 경간의 중앙에 가상 집중 횡하중을 작용시킨 후 이로 인한 지점들에서의 재단 moment를 계산한다. 이때 각 경간중앙의 가상 집중 횡하중 방향은 좌굴 mode를 고려하여 교호로 바뀌도록 한다. 제 3단계에서는 또 하나의 제안식과 재단 모멘트를 이용하여 Kinney의 부분 고정도를 결정한다. 최종단계에서는 부분 고정도를 이용하여 유효길이를 각 경간별로 산정한다. 제안한 방법은 다 경간 압축재에서 어느 경간이 맨 먼저 좌굴을 일으키는지 또한 이때의 임계하중은 어떤값을 갖는지를 예측하게 한다.

1. Introduction

The elastic critical load of a compression member is the most important factor to be considered in the structural design of that member. The critical load is usually expressed as a function of the effective length factor. The effective length factors (or K-factors) for a single span compression member with arbitrary loading and boundary conditions can be determined by analytical or numerical methods. For framed columns, the so-called alignment chart of AISC manual are commonly utilized for the determination of K-factors.

In the case of a continuous compression member, however, the critical load is difficult to determine. Furthermore, ti is not possible to predict the span that buckles first(or the span that governs the stability of the whole member). The difficulties compoynd with increasing numbers of span, as well as with different loading condition for each span. The sectional property change of each span makes the critical load determination more difficult. Structural engineers who are accustomed to the effective length factor concept have no means to determine the K-factor.

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In this paper, beam-analogy method is proposed for an easy determination of the effective length factor for each span of a continuous compression member. It can also predict the critical span that buckles first. The proposed method reveals near-exact critical load for any type of continuous compression member.

2. Beam-analogy method

The proposed method for the determination of the effective legnth factors of a continuous compression member can be divided into four steps, which is to be illustrated with a two-span continuous member in Fig. 1. The first step of the method is to replace the continuous compression member by a continuous beam with the same span lengths and sectional properties as shown in Fig. 2(a). Along this beam, a virtual concentrated load at each midspan is made to change its direction to simulate the buckling mode.

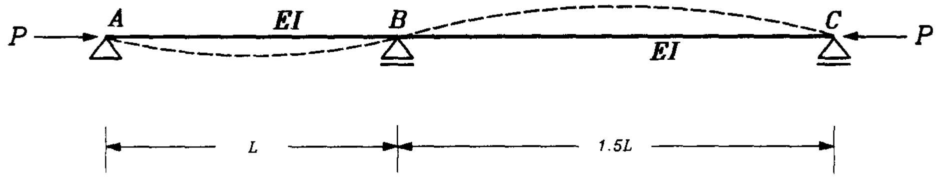


Fig. 1 Two-span continuous member

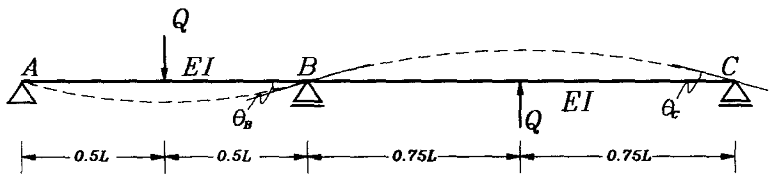
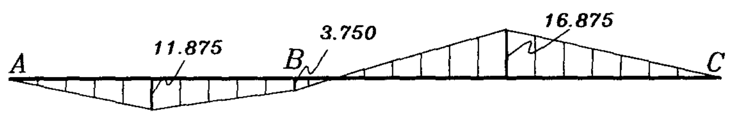


Fig. 2(a) A continuous beam



$\theta_A = 6.250$, $\theta_B = -7.500$, $\theta_C = 9.375$

Fig. 2(b) B.M.D ($\times QL/40$) and rotation angles ($\times QL^2/80EI$)

$$f_{AB}=0.0 \qquad f_{BA}=0.20, \quad f_{BC}=0.273 \qquad f_{CB}=0.0$$

$$(P_{cr})_{AB}=1.20\frac{\pi^2 EI}{L^2}, \qquad (P_{cr})_{BC}=1.273\frac{\pi^2 EI}{L^2}$$

Fig. 2(c) Fixity factors and critical loads

Fig. 2 An analogous beam

The second step is the structural analysis of the continuous beam in order to obtain the end moments and rotation angles at the supports. Stress analysis can be performed by any of several methods, among which the analysis results by the conventional slope-deflection method are shown in Fig.2 (b).

The third step is to determine the Kinney's fixity factors at the midsupports by using the following equation.

$$|M_\alpha| = \left(\frac{EI}{L}\right)_{\alpha\beta} \cdot \frac{f_{\alpha\beta}}{1-f_{\alpha\beta}} |\theta_\alpha| \qquad (1)$$

In the above, $|\ |$ denotes absolute value and Kinney's fixity, $f_{\alpha\beta}$ is zero ($f_{\alpha\beta}=0.0$) when the member end is simply supported and unity ($f_{\alpha\beta}=1.0$) when the end is completely fixed. When Eq.(1) is applied to midsupport, B of Fig. 2(b), one obtains

$$\frac{3.75}{40}QL=\frac{4EI}{L}\cdot\frac{f_{BA}}{1-f_{BA}}\cdot\frac{7.50QL^2}{80EI} \Rightarrow f_{AB}=0.200$$

$$\frac{3.75}{40}QL=\frac{4EI}{1.5L}\cdot\frac{f_{BC}}{1-f_{BC}}\cdot\frac{7.50QL^2}{80EI} \Rightarrow f_{AC}=0.273$$

The fixity factors at the exterior supports are zero ($f_{AB}=f_{CB}=0.0$) by the definition of fixity factor. The last step is the determination of the effective length factor of each span by using the following relationship;

$$(K)_{\alpha\beta}=((1+f_{\alpha\beta})\cdot(1+f_{\beta\alpha}))^{-0.5} \qquad (2)$$

When the above effective length factor is introduced, the critical load is expressed by

$$(P_{cr})_{\alpha\beta}=\left(\frac{\pi^2 EI}{(KL)^2}\right)_{\alpha\beta}=(1+f_{\alpha\beta})\cdot(1+f_{\beta\alpha})\cdot\pi^2\left(\frac{EI}{L^2}\right)_{\alpha\beta} \qquad (3)$$

When Eq. (2) and (3) are applied to the present problem, the critical load and K-factor for each span are given by

$$(K)_{AB} = ((1+0.0)\cdot(1+0.2))^{-0.5} = 0.913, \quad (P_{cr})_{AB} = \frac{\pi^2 EI}{(0.913L)^2} \doteqdot 11.840\frac{EI}{L^2}$$

$$(K)_{AB} = ((1+0.273)\cdot(1+0.0))^{-0.5} = 0.886, \ (P_{cr})_{BC} = \frac{\pi^2 EI}{(0.886\times 1.5L)^2} \doteqdot 5.588\frac{EI}{L^2} \quad (4)$$

From the above calculation, one can see that span BC governs the stability of the member. Fig.1 is the very example chosen by Chen's Structural stability textbook, where the neutral equilibrium method was applied to obtain $P_{cr} = 5.890EI/L^2$. When this is compared with Eq(4), an error of -5.2% is observed.

3. Comparisons of the critical loads

The critical load for the framed columns as well as the continuous compression members can be determined by the modified slope-deflection method. When axial force P is included in the derivation of the slope-deflection equations, one can obtain;

$$M_{\alpha\beta} = \left(\frac{EI}{L}\right)_{\alpha\beta}\cdot(\alpha_n\theta_A + \alpha_f\theta_B) \quad (5.\ a)$$

$$M_{\beta\alpha} = \left(\frac{EI}{L}\right)_{\alpha\beta}\cdot(\alpha_f\theta_A + \alpha_n\theta_B) \quad (5.\ b)$$

where the coefficients, α_n and α_f of the rotation angles θ_A and θ_B are so-called Merchants's stability functions defined by;

$$\alpha_n = \frac{\Phi_n}{\Phi_n^2 - \Phi_f^2}, \quad \alpha_f = \frac{\Phi_f}{\Phi_n^2 - \Phi_f^2} \quad (6.\ a,\ b)$$

with

$$kL = \sqrt{PL^2/EI}$$

$$\Phi_n = \frac{1}{(kL)^2}(1 - kL\cot kL), \quad \Phi_f = \frac{1}{(kL)^2}(kL\csc kL - 1) \quad (7.\ a,\ b)$$

The successive application of Eq.(5) to the continuous member of Fig. 1, the enforcement of boundary conditions at exterior ends and applying moment equilibrium condition at midsupport yield the following matrix equation;

$$\begin{vmatrix} \alpha_n & \alpha_f & 0 \\ \alpha_f & (\alpha_n + \alpha_{n1}/1.5) & \alpha_{f1}/1.5 \\ 0 & \alpha_{f1} & \alpha_{n1} \end{vmatrix} = \begin{pmatrix} \theta_A \\ \theta_B \\ \theta_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (8)$$

The above equation yields the following characteristic equation (α_{n1}, and α_{f1} are related to $k_1L = 1.5kL$)

$$det\begin{vmatrix} \alpha_n & \alpha_f & 0 \\ \alpha_f & (\alpha_n + \alpha_{n1}/1.5) & \alpha_{f1}/1.5 \\ 0 & \alpha_{f1} & \alpha_{n1} \end{vmatrix} = 0$$

or when expanded

$$\left(\alpha_n + \frac{\alpha_{n1}}{1.5}\right)\alpha_n \cdot \alpha_{f1} - \frac{\alpha_n}{1.5} \cdot \alpha_{f1}^2 - \alpha_{n1} \cdot \alpha_f^2 = 0 \tag{9}$$

The least root satisfying above equation can be found by the trial and error technique, which yields $kL = 2.4265$ and $P_{cr} \fallingdotseq 5.888EI/L^2$. Note that the value 5.888 coincides with Chen's result.

Nowadays numerical methods are more common for the solution of engineering problems. For example, the finite difference or finite element method can be used for the stability analysis of continuous members. In this study, the critical loads were also determined by the finite element method and Eq. (10) shows the element matrices.

$$[k] = [k_b] - [k_g] \tag{10}$$

$$[k_b] \text{ (=flexural stiffness matrix)} = \frac{EI(e)}{l^3}\begin{vmatrix} 12 & & & symm \\ -6l & 4l^2 & & \\ -12 & 6l & 12 & \\ -6l & 2l^2 & 6l & 4l^2 \end{vmatrix} \tag{10.a}$$

$$[k_g] \text{ (=geometric stiffness matrix)} = \frac{P}{30l}\begin{vmatrix} 36 & & & symm \\ -3l & 4l^2 & & \\ -36 & 3l & 36 & \\ -3l & -l^2 & 3l & 4l^2 \end{vmatrix} \tag{10.b}$$

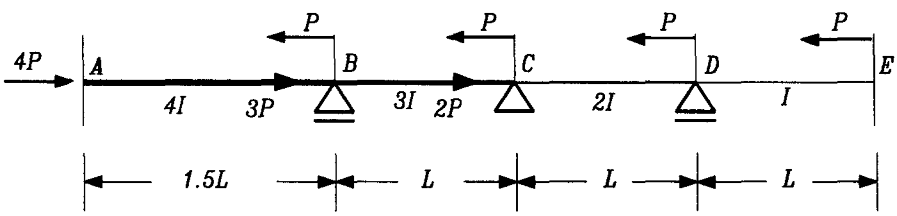
Above equations are derived by equating the flexural strain energy of the linear element having two degrees of freedom at each node to the work done by the constant axial P. As far as the final analysis results are concerned, the critical loads by the finite element method coincide with those determined by the modified slope-deflection method when the member is subdivided into finer elements.

4. Illustrative example

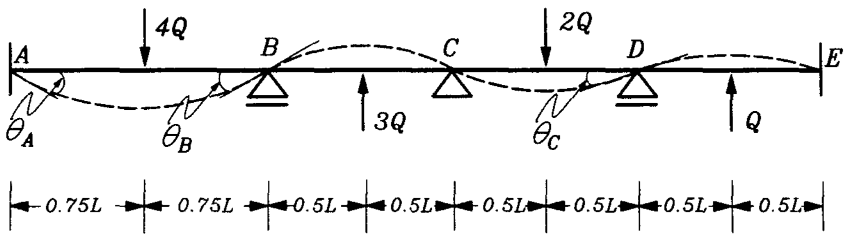
For a better understanding of the proposed method, another continuous compression member shown in Fig. 3(a), has been chosen. Fig. 3(b) shows the corresponding analogous beam. For the boundary conditions of the exterior ends, A and E, the following combinations are considered

① simple - simple ($f_{AB} = 0.0$, $f_{ED} = 0.0$) ② simple - fixed ($f_{AB} = 0.0$, $f_{ED} = 1.0$)

③ fixed - simple ($f_{AB} = 1.0$, $f_{ED} = 0.0$) ④ fixed - fixed ($f_{AB} = 1.0$, $f_{ED} = 1.0$)



(a) Continuous compression member



(b) Continuous beam under virtual loads

Fig. 3 Four-span continuous member

The stress analysis results for each boundary condition are summarized in Table (1). In the table, the coefficient, c_2 for M_α ($\alpha = A, B, \cdots, E$) denotes the absolute value of end moment.

Table 1. Rotation angles and absolute end moment coefficients of analogous beam

$$\theta_\alpha = C_1 \cdot QL^2/8EI\,, \quad M_\alpha = C_2 \cdot QL/8$$

	θ_A	θ_B	θ_C	θ_D	θ_E	M_A	M_B	M_C	M_D	M_E
① S-S	0.951	-0.777	0.590	-0.533	0.516	0.0	2.784	0.588	0.096	0.0
② S-F	0.948	-0.771	0.569	-0.440	0.0	0.0	2.832	0.796	0.756	1.880
③ F-S	0.0	-0.532	0.510	-0.504	0.502	8.837	0.325	0.068	0.010	0.0
④ F-F	0.0	-0.527	0.491	-0.414	0.0	8.106	0.379	0.272	0.654	1.827

Table 2. Fixity factors at the member supports

$$|M_\alpha| = \left(\frac{4EI}{L}\right)_{\alpha\beta} \cdot \frac{f_{\alpha\beta}}{1 - f_{\alpha\beta}} \cdot |\theta_\alpha|$$

	A	B		C		D		E
	f_{AB}	f_{BA}	f_{BC}	f_{CB}	f_{CD}	f_{DC}	f_{DE}	f_{ED}
① S-S	0.0	0.251	0.230	0.076	0.111	0.022	0.043	0.0
② S-F	0.0	0.256	0.234	0.104	0.149	0.177	0.300	1.0
③ F-S	1.0	0.054	0.048	0.011	0.016	0.002	0.005	0.0
④ F-F	1.0	0.063	0.056	0.044	0.065	0.165	0.283	1.0

Table 3. Elastic critical load coefficient, C s for the member

$$(P_{cr})_{\alpha\beta} = (1 + f_{\alpha\beta})(1 + f_{\beta\alpha})\left(\frac{\pi}{L}\right)^2 EI = C \cdot \frac{EI}{L^2}$$

	Each span as an isolated member				system as a whole	S.D.M	F.E.M	error(%)
	AB	BC	CD	DE				
① S-S	5.487	13.062	11.206	10.294	5.487	5.858	5.858	-6.3
② S-F	5.509	13.445	13.347	25.661	5.509	5.875	5.875	-6.2
③ F-S	9.247	10.457	10.457	9.919	9.247	9.484	9.484	-2.5
④ F-F	9.325	10.881	12.245	25.325	9.325	9.737	9.737	-4.2

Table (2) shows the Kinney's fixity factor determined by Eq. (2). For example f_{AB} and f_{BC} for the first case of boundary conditions (① S-S) are determined in the following way (see Table. (1));

$$2.784 \cdot \frac{QL}{8} = \frac{4\times 4EI}{1.5L} \cdot \frac{f_{BA}}{1-f_{BA}} \cdot \frac{0.777QL^2}{8EI}, \qquad f_{BA} \fallingdotseq 0.251$$

$$2.784 \cdot \frac{QL}{8} = \frac{4\times 3EI}{L} \cdot \frac{f_{BC}}{1-f_{BC}} \cdot \frac{0.777QL^2}{8EI}, \qquad f_{BC} \fallingdotseq 0.230$$

The critical load of each span as an isolated member is determined by using Eq. (3). For example if one choose span AB and BC for boundary conditions (④F-F), one obtains;

$$(4P_{cr})_{AB} = (1.0+1.0)(1.0+1.063)\left(\frac{\pi}{1.5L}\right)^2 \cdot (4EI) \quad \Rightarrow \quad (P_{cr})_{AB} \fallingdotseq 9.325\frac{EI}{L^2}$$

$$(3P_{cr})_{BC} = (1.056)(1.044)\left(\frac{\pi}{L}\right)^2 \cdot (3EI) \quad \Rightarrow \quad (P_{cr})_{BC} \fallingdotseq 10.881\frac{EI}{L^2}$$

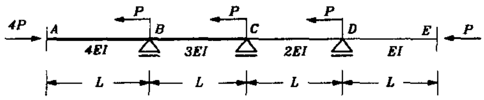
Among the loads for several spans, the least value is the very critical load that governs the stability of the given continuous member. The span AB governs the stability of the member for all cases of the boundary conditions. In the table, columns SDM and FEM denote critical loads determined by the modified slope-deflection and finite element method, respectively. It is observed that both methods yield the same critical load coefficient. It should also be noted that the proposed method gives lower bound errors for all cases of the boundary conditions, which proves the validity of applying the proposed methods to practical structural design. The present study focuses on the K-factors for the continuous member of Fig. 3, which are easily obtained from Table 2. The final results are summarized in Table 4.

Table 4. K-factors for each span

B.C \ SPAN	AB		BC		CD		DE	
① S-S	f_{AB}=0.0, f_{BA}=0.251	0.894	f_{BC}=0.230, f_{CB}=0.076	0.869	f_{CD}=0.111, f_{DC}=0.022	0.938	f_{DE}=0.043, f_{ED}=0.0	0.979
② S-F	f_{AB}=0.0, f_{BA}=0.256	0.892	f_{BC}=0.234, f_{CB}=0.104	0.857	f_{CD}=0.149, f_{DC}=0.177	0.860	f_{DE}=0.300, f_{ED}=1.0	0.620
③ F-S	f_{AB}=1.0, f_{BA}=0.054	0.689	f_{BC}=0.048, f_{CB}=0.011	0.971	f_{CD}=0.016, f_{DC}=0.002	0.991	f_{DE}=0.005, f_{ED}=0.0	0.997
④ F-F	f_{AB}=1.0, f_{BA}=0.063	0.686	f_{BC}=0.056, f_{CB}=0.044	0.952	f_{CD}=0.065, f_{DC}=0.165	0.898	f_{DE}=0.283, f_{ED}=1.0	0.624

As can be seen in Table (5), however, the stability governing span can be different if the stability governing parameters are changed.

Table 5. Critical load coefficients, C s four-span continuous member ($P_{cr} = CEI/L^2$)

	B.C	S.D.M	F.E.M	Proposed	error(%)
	$f_A=0.0/f_E=0.0$	9.869	9.869	9.869(ALL)	0.0
	$f_A=0.0/f_E=1.0$	10.213	10.213	9.938(AB)	-2.7
	$f_A=1.0/f_E=0.0$	11.244	11.244	10.136(DE)	-9.8
	$f_A=1.0/f_E=1.0$	12.613	12.613	11.787(CD)	-6.6

In the fourth column, the proposed method, (ALL) denotes the simultaneous buckling of all spans while (CD) indicates that the span CD governs the stability of the whole member.

5. Conclusions

A new method using the Kinney's fixity factors at the member supports has been developed for the determination of the effective length factors of continuous (or multi-span) compression members. The fixity factors are determined by using the stress analysis results of a continuous beam having the same dimensions as the compression member and is subjected to a lateral concentrated load at the center of each span. The analysis results of the continuous compression member by the proposed method lead to the following conclusions:

- The effective length factor and critical load of each span as an isolated member are easy to determine.
- Prediction of the span that buckles first under any loading and boundary conditions is possible.
- The critical load governing the stability of the continuous compression member as a whole is less (max 10%) than that determined by other methods.

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