BTZ black hole and warped product spacetimes

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Abstract

Exploiting a multiply warped products manifold scheme, we study the interior solutions of the (2+1) Banados-Teitelboim-Zanelli black holes and the exterior solutions of the (2+1) de Sitter black holes.

1 INTRODUCTION

Since the pioneering work in 1976 [1] thermal Hawking effects on a curved manifold [2] have been studied as an Unruh effect in a higher flat dimensional space-time. Following the global embedding Minkowski space approach [3, 4, 5, 6], several authors recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds in (2+1) dimensions [7, 8, 9, 10, 11, 12, 13] and in (3+1) dimensions [7, 14, 15, 16]. However all these higher dimensional embedding solutions have been constructed outside the event horizons of the metrics.

On the other hand, the concept of a warped product manifold was introduced [17] to provide a class of complete Riemannian manifolds with everywhere negative curvature [18], and was developed to point out that several of the well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products [19]. Furthermore, certain causal and completeness properties of a spacetime could be determined by the presence of a warped product structure [20], and general theory of warped products were applied to discuss the special cases of Robertson-Walker and Schwarzshild manifold. The role of warped products in the study of exact solutions to Einstein's equations is now firmly established to generate interest in other areas of geometry. Recently, the warped product scheme has been applied to higher dimensional theories such as the Randall-Sundrum model [21, 22, 23] in five dimension and the non-singular warped Kaluza-Klein embeddings [24] in five to seven dimensional gauged supergravity theories. Moreover, the warped product scheme was applied to investigate warping functions associated with constant scalar curvature on globally null manifold [25]. Assuming the four dimensional spacetime to be a warped product of two surfaces, the four dimensional Einstein equations were also reduced to two dimensional ones to describe wormwholes and domainwalls of curvature singularities [26].

In order to investigate physical properties inside the black hole horizons, we briefly review a multiply warped product manifold $(M = B \times F_1 \times ... \times F_n, g)$ which consists of the Riemannian base manifold (B, g_B) and fibers (F_i, g_i) $(i = 1, ..., n)$ associated with the Lorentzian metric,

$$g = \pi_B^* g_B + \sum_{i=1}^{n} (f_i \circ \pi_B)^2 \pi_i^* g_i \tag{1.1}$$

where π_B, π_i are the natural projections of $B \times F_1 \times ... \times F_n$ onto B and F_i, respectively, and f_i are positive warping functions. For the specific case of $(B = R, g_B = -d\mu^2)$, the above metric is rewritten as

$$g = -d\mu^2 + \sum_{i=1}^{n} f_i^2 g_i, \tag{1.2}$$

to extend the warped product spaces to richer class of spaces involving multiply products. Moreover, the conditions of spacelike boundaries in the multiply warped product spacetimes [27] were also studied [28] and the curvature of the multiply warped product with C^0-warping functions was later investigated [29]. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of spacetime such as the Robertson-Walker manifold and the intermediate zone of RN manifold [30, 31]. Very recently, the interior Schwarzschild spacetime has been represented as a multiply warped product spacetime with warping functions [29] to yield the Ricci curvature in terms of f_1 and f_2 for the multiply warped products of the form $M = R \times_{f_1} R \times_{f_2} S^2$.

In this paper we will analyze the multiply warped product manifold associated with the charge black holes such as the Banados-Teitelboim-Zanelli (BTZ) and de Sitter (dS) metrics to investigate the physical properties inside the event horizons. We will exploit the multiply warped product scheme to investigate the interior solutions in (2+1) charged BTZ black holes in section 2, in (2+1) charged dS black holes in section 3 so that we can explicitly obtain the Ricci and Einstein curvatures inside the event horizons of these metrics.

2 BTZ BLACK HOLES

2.1 STATIC BTZ CASE

In order to investigate a multiply warped product manifold for the static BTZ interior solution, we start with the three-metric inside the horizon

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 d\phi^2 \tag{2.1}$$

with the lapse function for the interior solution

$$N^2 = m - \frac{r^2}{l^2}. \tag{2.2}$$

Note the event horizon r_H is given by $r_H = m^{1/2}l$. Furthermore the lapse function can be rewritten in terms of the event horizon as follows

$$N^2 = \frac{(r_H + r)(r_H - r)}{l^2} \tag{2.3}$$

which is well defined in the region $r < r_H$. Now we define a new coordinate μ as follows

$$d\mu^2 = N^{-2}dr^2, \tag{2.4}$$

which can be integrated to yield

$$\mu = \int_0^r dx \frac{l}{[(r_H + x)(r_H - x)]^{1/2}}, \tag{2.5}$$

whose analytic solution is of the form

$$\mu = l \sin^{-1}\left(\frac{r}{r_H}\right) = F(r). \tag{2.6}$$

Moreover, we have the following boundary conditions

$$\lim_{r\to r_H} F(r) = \frac{l\pi}{2}, \quad \lim_{r\to 0} F(r) = 0, \tag{2.7}$$

and $dr/d\mu > 0$ implies F^{-1} is well-defined function. Exploiting the above new coordinate (2.6), we rewrite the metric (2.1) as a warped products

$$ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu) d\phi^2 \tag{2.8}$$

where

$$\begin{aligned} f_1(\mu) &= \left(m - \frac{F^{-2}(\mu)}{l^2}\right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu). \end{aligned} \tag{2.9}$$

After some algebra, we obtain the following nonvanishing Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{f_1''}{f_1} - \frac{f_2''}{f_2}, \\ R_{tt} &= \frac{f_1 f_1' f_2'}{f_2} + f_1 f_1'', \\ R_{\phi\phi} &= \frac{f_1' f_2 f_2'}{f_1} + f_2 f_2''. \end{aligned} \tag{2.10}$$

Using the explicit expressions for f_1 and f_2 in (2.9), one can obtain identities for f_1, f_1' and f_1'' in terms of f_1, f_2 and their derivatives

$$\begin{aligned} f_1 &= f_2', \\ f_1' &= -\frac{f_2}{l^2}, \\ f_1'' &= \frac{f_1 f_1'}{f_2}, \end{aligned} \tag{2.11}$$

to yield the Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{2f_1'}{f_2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \tag{2.12}$$

and the Einstein scalar curvature

$$R = -\frac{6}{l^2}, \tag{2.13}$$

in the interior of the static BTZ black hole horizon.

2.2 CHARGED BTZ CASE

Now we consider a multiply warped product manifold associated with the charged BTZ three-metric (2.1) inside the horizon with the charged lapse function [10]

$$N^2 = m - \frac{r^2}{l^2} + 2Q^2 \ln\ r. \tag{2.14}$$

Note the event horizon r_H satisfies the equation $0 = m - \frac{r_H^2}{l^2} + 2Q^2 \ln\ r_H$, and for the range $Ql < r < r_H$ we have the coordinate μ in Eq. (2.4) as follows

$$\mu = \int_{Ql}^{r} dx\ \frac{l}{(m - \frac{r^2}{l^2} + 2Q^2 \ln\ r)^{1/2}}. \tag{2.15}$$

Note that $dr/d\mu > 0$ implies F^{-1} is well-defined function. Exploiting the above coordinate (2.15), we can obtain the warped products (2.8) with the modified f_1 and f_2 as below

$$\begin{aligned} f_1(\mu) &= \left(m - \frac{F^{-2}(\mu)}{l^2} + 2Q^2 \ln\ F^{-1}(\mu)\right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \tag{2.16}$$

to yield the Ricci curvature components

$$\begin{aligned} R_{\mu\mu} &= -\frac{2f_1'}{f_2} + \frac{2Q^2}{f_2^2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2} - \frac{2Q^2 f_1^2}{f_2^2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \tag{2.17}$$

and the Einstein scalar curvature

$$R = -\frac{6}{l^2} + \frac{2Q^2}{f_2^2}, \tag{2.18}$$

in the interior of the charged BTZ black hole horizon. Now it seems appropriate to comment on the relations between the interior and exterior solutions in the charged BTZ black hole. In the outside the event horizon r_H where the three-metric is given by

$$ds^2 = -(-m+\frac{r^2}{l^2}-2Q^2 \ln\ r)^2 dt^2+(-m+\frac{r^2}{l^2}-2Q^2 \ln\ r)^{-2} dr^2+r^2 d\phi^2, \tag{2.19}$$

one can obtain the Ricci curvature components in terms of the warping functions f_1 and f_2 as follows

$$\begin{aligned} R_{rr} &= -\frac{2f_1'}{f_1^2 f_2} + \frac{2Q^2}{f_1^2 f_2^2}, \\ R_{tt} &= \frac{2f_1^2 f_1'}{f_2} - \frac{2Q^2 f_1^2}{f_2^2}, \\ R_{\phi\phi} &= 2f_2 f_1', \end{aligned} \tag{2.20}$$

and the Einstein scalar curvature identical to the interior case (2.18). Here one notes that the Ricci components R_{tt} and $R_{\phi\phi}$ are the same as those of interior case.

Moreover from the definition of the coordinate μ in Eq. (2.4) one can obtain the identity

$$R_{\mu\mu} = f_1^2 R_{rr} \tag{2.21}$$

which is also attainable from the Ricci components $R_{\mu\mu}$ and R_{rr} in Eqs. (2.17) and (2.20). One can thus show that all the Ricci components and the Einstein scalar curvature are identical both in the exterior and interior of the event horizon r_H without discontinuities.

2.3 ROTATING BTZ CASE

Now we consider a multiply warped product manifold associated with the rotating BTZ black hole inside the horizon whose three-metric is given by

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{2.22}$$

where the lapse and shift functions are given by

$$\begin{aligned} N^2 &= m - \frac{r^2}{l^2} - \frac{J^2}{4r^2}, \\ N^\phi &= -\frac{J}{2r^2}, \end{aligned} \tag{2.23}$$

with an angular momentum J. Note the event horizon $r_\pm$ satisfies the equation $0 = m - \frac{r_\pm^2}{l^2} - \frac{J^2}{4r_\pm^2}$ to yield the lapse function in terms of the event horizons as follows

$$N^2 = \frac{(r_+^2 - r^2)(r^2 - r_-^2)}{r^2 l^2} \tag{2.24}$$

which, for the interior solution, is well defined in the region $r_- < r < r_+$.

Now we define a new coordinate μ as in Eq. (2.4) to yield

$$\mu = \int_{r_-}^{r} dx \, \frac{l}{(m - \frac{r^2}{l^2} - \frac{J^2}{4r^2})^{1/2}}, \tag{2.25}$$

whose analytic solution is of the form

$$\mu = l \sin^{-1}\left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right)^{1/2} = F(r). \tag{2.26}$$

Moreover, we have the following boundary conditions

$$\lim_{r \to r_+} F(r) = \frac{l\pi}{2}, \quad \lim_{r \to r_-} F(r) = 0, \tag{2.27}$$

and $dr/d\mu > 0$ implies F^{-1} is well-defined function. Note that $dr/d\mu > 0$ implies F^{-1} is well-defined function and in the vanishing angular momentum limit $J \to 0$, the above solution (2.26) reduces to the static BTZ case (2.6).

Exploiting the above new coordinate (2.26), we can obtain

$$ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu)(d\phi + N^\phi dt)^2 \tag{2.28}$$

to yield the metric of the warped product form (2.8) in a comoving coordinates where one can replace[1] $d\phi + N^\phi dt \to d\phi$ to obtain the modified f_1 and f_2 as below

$$\begin{aligned} f_1(\mu) &= \left(m - \frac{F^{-2}(\mu)}{l^2} - \frac{J^2}{4F^{-2}(\mu)}\right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \tag{2.29}$$

[1]Here one notes that the detector locates in the comoving coordinates with the angular velocity $\dot{\phi} = d\phi/dt = -g_{t\phi}/g_{\phi\phi} = -N^\phi$.

and the Ricci curvature components

$$
\begin{aligned}
R_{\mu\mu} &= -\frac{2f_1'}{f_2}+\frac{J^2}{f_2^4}, \\
R_{tt} &= \frac{2f_1^2 f_1'}{f_2}-\frac{J^2 f_1^2}{f_2^4}, \\
R_{\phi\phi} &= 2f_2 f_1'.
\end{aligned} \tag{2.30}
$$

Here one notes that there does not exist an additional term associated with the angular momentum J in the $R_{\phi\phi}$ component since we have used the comoving coordinates. The Einstein scalar curvature is then given by

$$R=-\frac{6}{l^2}-\frac{J^2}{2f_2^4}, \tag{2.31}$$

in the interior of the charged BTZ black hole horizons. Note that in the $J\to 0$ limit, the above Ricci components (2.30) and Einstein scalar curvature (2.31) reduce to the corresponding ones in the static BTZ case.

3 dS BLACK HOLES

3.1 STATIC dS CASE

In order to investigate a multiply warped product manifold for the static dS exterior solution, we start with the three-metric (2.1) outside the horizon with the lapse function for the exterior solution

$$N^2=-m+\frac{r^2}{l^2}. \tag{3.1}$$

Note the event horizon r_H is given by $r_H=m^{1/2}l$. Furthermore the lapse function can be rewritten in terms of the event horizon as follows

$$N^2=\frac{(r+r_H)(r-r_H)}{l^2} \tag{3.2}$$

which is well defined in the region $r > r_H$. Now we define a new coordinate μ as in the BTZ case to yield

$$\mu = l \cosh^{-1}\left(\frac{r}{r_H}\right) = F(r), \tag{3.3}$$

and the boundary condition

$$\lim_{r \to r_H} F(r) = 0, \tag{3.4}$$

and $dr/d\mu > 0$ implies F^{-1} is well-defined function. Exploiting the above new coordinate (3.3), we rewrite the metric (2.1) with the lapse function (3.1) as a warped products (2.8) where

$$\begin{aligned} f_1(\mu) &= \left(-m + \frac{F^{-2}(\mu)}{l^2}\right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \tag{3.5}$$

to yield, in the exterior of the static dS black hole horizon, the same form of Ricci curvature components (2.12) as those of the static BTZ case, and the Einstein scalar curvature

$$R = \frac{6}{l^2}, \tag{3.6}$$

which has the opposite sign of the static BTZ result (2.13).

3.2 CHARGED dS CASE

Now we consider a multiply warped product manifold associated with the charged dS three-metric (2.1) outside the horizon with the charged lapse function

$$N^2 = -m + \frac{r^2}{l^2} + 2Q^2 \ln\, r. \tag{3.7}$$

Note the event horizon r_H satisfies the equation $0 = -m + \frac{r_H^2}{l^2} + 2Q^2 \ln\ r_H$, and for the range $r > r_H$ we have the coordinate μ

$$\mu = \int_{r_H}^{r} dx \frac{l}{(-m + \frac{r^2}{l^2} + 2Q^2 \ln\ r)^{1/2}}. \tag{3.8}$$

Note that $dr/d\mu > 0$ implies F^{-1} is well-defined function. Exploiting the above coordinate (2.15), we can obtain the warped products (2.8) with the modified f_1 and f_2 as below

$$\begin{aligned} f_1(\mu) &= \left(-m + \frac{F^{-2}(\mu)}{l^2} + 2Q^2 \ln\ F^{-1}(\mu)\right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \tag{3.9}$$

to yield, in the exterior of the charged dS black hole horizon, the same form of Ricci curvature components (2.17) as those of the charged BTZ case, and the Einstein scalar curvature

$$R = \frac{6}{l^2} + \frac{2Q^2}{f_2^2}. \tag{3.10}$$

3.3 ROTATING dS CASE

Now we consider a multiply warped product manifold associated with the rotating dS black hole outside the horizon whose three-metric is given by (2.22) where the lapse and shift functions are now given by

$$\begin{aligned} N^2 &= -m + \frac{r^2}{l^2} - \frac{J^2}{4r^2}, \\ N^\phi &= -\frac{J}{2r^2}. \end{aligned} \tag{3.11}$$

Note the event horizon $r_\pm$ satisfies the equation $0 = -m + \frac{r_\pm^2}{l^2} - \frac{J^2}{4r_\pm^2}$ to yield the lapse function in terms of the event horizons as follows

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2} \tag{3.12}$$

which, for the exterior solution, is well defined in the region $r > r_+$.

Now we define a new coordinate μ as in Eq. (2.4) to yield

$$\mu = \int_{r_+}^{r} dx \, \frac{l}{(-m + \frac{r^2}{l^2} - \frac{J^2}{4r^2})^{1/2}}, \tag{3.13}$$

whose analytic solution is of the form

$$\mu = l \cosh^{-1} \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} = F(r). \tag{3.14}$$

Moreover, we have the following boundary conditions

$$\lim_{r \to r_+} F(r) = 0, \tag{3.15}$$

and $dr/d\mu > 0$ implies F^{-1} is well-defined function. Note that $dr/d\mu > 0$ implies F^{-1} is well-defined function and in the vanishing angular momentum limit $J \to 0$, the above solution (3.14) reduces to the static dS case (3.3). Exploiting the above new coordinate (3.14), we can obtain the metric (2.28) to yield the warped products (2.8) in a comoving coordinates where one can replace $d\phi + N^\phi dt \to d\phi$ and the modified f_1 and f_2 are given as below

$$\begin{aligned} f_1(\mu) &= \left(-m + \frac{F^{-2}(\mu)}{l^2} - \frac{J^2}{4F^{-2}(\mu)} \right)^{1/2}, \\ f_2(\mu) &= F^{-1}(\mu), \end{aligned} \tag{3.16}$$

to yield, in the exterior of the rotating dS black hole horizon, the same Ricci curvature components (2.30) as those of the rotating BTZ case, and the Einstein scalar curvature

$$R = \frac{6}{l^2} - \frac{J^2}{2f_2^4}. \tag{3.17}$$

Note that in the $J \to 0$ limit, the above Einstein scalar curvature (3.6) reduce to the corresponding ones in the static dS case.

4 CONCLUSIONS

We have studied a multiply warped product manifold associated with the BTZ (de Sitter) black holes to evaluate the Ricci curvature components inside (outside) the black hole horizons. Moreover, we have shown that all the Ricci components and the Einstein scalar curvatures are identical both in the exterior and interior of the event horizons without discontinuities for both the BTZ and dS black holes.

ACKNOWLEDGMENTS

STH, JC and YJP would like to acknowledge financial support in part from Korea Science and Engineering Foundation Grant (R01-2000-00015), (R01-2001-000-00003-0), and Ministry of Education, BK21 Project (D-1099), respectively.

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