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# Fuzzy Hyperpsaces : Fuzzy Compactness

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## Abstract

First, we investigate some properties of fuzzy compactness. Second, we introduce the concept of fuzzy local compactness in fuzzy topological space and study some of its properties. Finally, we investigate some relations between F-compactness in fuzzy topological spaces and one in fuzzy hyperspaces.

**Key words and phrases** : fuzzy compactness, normalized fuzzy topological space, fuzzy separation axioms, fuzzy local compactness, fuzzy hyperspace.

## 1. Introduction and preliminaries

Many authors [1-6,10-18] have investigated various properties in fuzzy sets and fuzzy topological spaces. In 2000, K.Hur, J.R.Moon and J.H.Ryou[7] introduced the concept of fuzzy hyperspace and studied some of its properties. Recently, K.Hur, C.J.Rhee and J.H. Ryou[8,9] have investigated some fundamental properties of fuzzy hyperspaces and some relations among fuzzy separation axioms in fuzzy topological spaces and fuzzy hyper-spaces. Also, Y.S.Ahn, K.Hur and J.H.Ryou[1] have studied some properties of fuzzy set-valued mappings.

In this paper, we deal with the followings: First, we list some concepts and results needed in the later sections. Second, we mainly list the concept of the fuzzy compactness in the sense of S.Ganguly and S.Saha[6] and some of its properties and obtain another some result, in particular, "Alexander Subbase Theorem".

**Definition 1.1[2].** Let $X$ be a nonempty set. Then a fuzzy set $A$ in $X$ is called :

(1) *an upper fuzzy set* if $A(x) > \frac{1}{2}$ whenever $A(x) \neq 0$ for each $x \in X$.

(2) *a lower fuzzy set* if $A(x) < \frac{1}{2}$ whenever $A(x) \neq 1$ for each $x \in X$.

It is clear that the only fuzzy sets in $X$ which are both upper and lower fuzzy sets are $\emptyset$ and $X$.

**Definition 1.2.** A fts $X$ is said to be :

(1) **fuzzy** $T_1$ (in short, $FT_1$)[5] if for any two fuzzy points $x_\lambda$ and $y_\mu$ in $X$ ;

(case 1) When $x \neq y$, there exist $U, V \in FO(X)$ such that $x_\lambda \in U$, $y_\mu \bar{q} U$ and $y_\mu \in V$, $x_\lambda \bar{q} V$.

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(case 2) When $x=y$ and $\lambda<\mu$(say), there exists a $U\in FO(X)$ such that $y_\mu qU$ and $x_\lambda \bar{q} U$.

(2) $FT_{1w}$[3] if for any two distinct fuzzy points $x_\lambda$ and $y_\mu$ in $X$, there exist $U, V\in FO(X)$ such that $x_\lambda\in U\subset y_\mu^c$ and $y_\mu\in V\subset x_\lambda^c$.

**Definition 1.3[3].** A fts $X$ is said to be $FT_{2w}$ if for any two distinct fuzzy points $x_\lambda$ and $y_\mu$ in $X$, there exist $U, V\in FO(X)$ such that $x_\lambda\in U$, $y_\mu\in V$ and $U\odot V=\emptyset$.

It is clear that if $X$ is $FT_2$ then it is $FT_{2w}$ and every $FT_{2w}$-space is a $FT_{1w}$-space.

**Definition 1.4[3].** A fts $X$ is said to be $FR$ (ii) if for each $x_\lambda\in F_p(X)$ and each $B\in FC(X)$ such that $x_\lambda\in B^c$, there exist $U, V\in FO(X)$ such that $x_\lambda\in U$, $B\subset V$ and $U\odot V=\emptyset$.

A $T_{1w}FR$ (ii)-space is called a $FT_{3w}$*-space.*

**Definition 1.5[3].** A fts $X$ is said to be $FN$ (ii) if for any two $A, B\in FC(X)$ such that $A\odot B=\emptyset$, there exist $U, V\in FO(X)$ such that $A\subset U\subset V^c\subset B^c$.

A $FT_{1w}FN$ (ii)-space is called a $FT_{4w}$*-space.*

**Result 1.A[3, Theorem 3.10].** Let $X$ be a fts. Then the following are equivalent:

(1) $X$ is $FR$ (ii).

(2) For each $x_\lambda\in F_p(X)$ and each $G\in FO(X)$ such that $x_\lambda\in G$, there exists a $U\in FO(X)$ such that $x_\lambda\in U\subset \mathrm{cl}\, U\subset G$.

**Notations 1.6.** For a fts $X$, let

$I_0^X=\{E: E$ is a nonempty F-closed set in $X\}$,

$I_0^A=\{E\in I_0^X: E\subset A\}$, where $A\in I^X$,

$K(X)=\{E\in I_0^X: E$ is F-compact in $X\}$.

**Result 1.B[8, Lemma 3.4].** Let $X$ be a $FT_1$-space and let $A\in I^X$. Then :

(1) $\mathrm{cl}\, I_0^A\subset I_0^{\,\mathrm{cl}A}$.

(1′) If $X$ is a $(q,\in)$-fts, then

$\mathrm{cl}\, I_0^A=I_0^{\,\mathrm{cl}A}$, i.e., $I_0^{\,\mathrm{cl}A}\subset \mathrm{cl}\, I_0^A$.

(2) $\mathrm{int}\, I_0^A=I_0^{\,\mathrm{int}A}$.

**Result 1.C[8, Theorem 1.18].** Let $X$ be a $(q,\in)$-fts and let $U_i\in I^X$ for each $i=1,\cdots,n$. If $X$ is $FT_1$-space, then

$\mathrm{cl}\langle U_1,\cdots,U_n\rangle=\langle \mathrm{cl}\, U_1,\cdots,\mathrm{cl}\, U_n\rangle$.

**Definition 1.7[8].** A fuzzy point $x_\lambda$ in a set $X$ is called:

(1) an *upper fuzzy point* if $\lambda>\frac{1}{2}$.

(2) a *lower fuzzy point* if $\lambda<\frac{1}{2}$.

**Result 1.D[9, Theorem 3.4].** Let $X$ be a $FT_1$-space such that each F-closed set in $X$ have an upper fuzzy point. If $X$ is $FR$ (ii) if and only if $I_0^X$ is Hausdorff.

## 2. Compactness in fuzzy topological spaces

**Definition 2.1.** A fts $X$ is said to be *normalized* if for each $x_\lambda\in F_p(X)$, there exists $U\in FO(X)$ such that $U(x)=1$.

**Lemma 2.2.** Let $X$ be a normalized $FT_{2w}$-space, let $A$ a F-compact set in $X$ and let $y_\mu\in A^c$. Then there exist $U, V\in FO(X)$ such that $A\subset U$, $y_\mu\in V$, $V(y)=1$ and $U\odot V=\emptyset$.

**Definition 2.3.** Let $X$ be a fts and let $\mathscr{A} \subset I^X$. Then :

(1) $\mathscr{A}$ is said to be *inadequate* if it does not cover $C_\lambda(0<\lambda\leq 1)$, i.e., $(\bigcup \mathscr{A})(x) \neq 1$ for each $x \in X$.

(2) $\mathscr{A}$ is said to be *finitely inadequate* if no finite subcollection covers $C_\lambda$.

**Theorem 2.4(Alexander Subbase Theorem).** Let $X$ be a fts and let $\mathscr{S}$ a subbase for $X$. Then $X$ is F-compact if and only if every cover of $C_\lambda(0<\lambda\leq 1)$ by members of $\mathscr{S}$ has a finite subcover.

## 3. Local compactness in fuzzy topological spaces

**Definition 3.1.** A fts $X$ is said to be *fuzzy locally compact*(in short, *locally F-compact*) at $x_\lambda \in F_p(X)$ if there exists a F-open set $U$ and a F-compact set $K$ in $X$ such that $x_\lambda \in U \subset K$. A fts $X$ is said to be *locally F-compact* if it is locally F-compact at each of its fuzzy points.

It is clear that every compact fts is locally compact.

**Theorem 3.2.** Let $X$ be a normalized $FT_{2w}$-space and let $x_\lambda \in F_p(X)$. Then $X$ is locally F-compact at $x_\lambda$ if and only if for each neighborhood $V$ of $x_\lambda$ there exists a F-open set $U$ and a F-compact set $K$ in $X$ such that $x_\lambda \in U \subset V$ and $U \subset K \subset \mathrm{cl}\,U$.

**Theorem 3.3.** Let $X$ be a normalized $FT_{2w}$-space and let $x_\lambda \in F_p(X)$. Then $X$ is locally F-compact at $x_\lambda$ if and only if there exists a neighborhood $U$ of $x_\lambda$ such that $\mathrm{cl}\,U$ is F-compact in $X$.

**Corollary 3.3.** A normalized $FT_{2w}$-space $X$ is locally F-compact if and only if for each $x_\lambda \in F_p(X)$ and each neighborhood $V$ of $x_\lambda$, there exists a neighborhood $U$ of $x_\lambda$ such that $\mathrm{cl}\,U$ is F-compact in $X$ and $\mathrm{cl}\,U \subset V$.

**Theorem 3.4.** Every F-closed subspace of a normalized locally compact $FT_{2w}$-space is locally F-compact.

**Theorem 3.5.** Let $X$ be a locally F-compact fts, let $Y$ a fts and let $f: X \to Y$ an F-open continuous surjection. Then $(Y, T_Y)$ is locally F-compact.

## 4. Compactness and local compactness in fuzzy hyperspaces

**Definition 4.1.** A fts $X$ is said to be *finitely F-compact* if each finite fuzzy set in $X$ is F-compact in $X$.

**Theorem 4.2.** Let $X$ be an upper finitely F-compact $FT_1$-space. If $\mathscr{U}$ is open in the subspace $K(X)$, then $\bigcup \mathscr{U} \in FO(X)$.

**Theorem 4.3.** Let $X$ be a fts. Then $\bigcup \mathscr{B} \in K(X)$ for each $\mathscr{U} \in K(K(X))$.

**Theorem 4.4.** Let $X$ be a normalized fts. If $X$ is $FT_{2w}$, then $K(X)$ is Hausdorff.

**Theorem 4.5.** Let $X$ be a $(q, \in)-FT_1$-space. If $I_0^X$ is compact, then $X$ is F-compact.

**Theorem 4.6.** Let $X$ be a normalized $(q, \in)-FT_1$-space. If $X$ is $FR$ (ii), then $K(X)$ is regular.

**Theorem 4.7.** Let $X$ be a normalized $FT_1$-space. If $I_0^X$ is compact, then $X$ is F-compact.

**Theorem 4.8.** Let $X$ be a $(q, \in)-FT_1$-space and let each F-closed set in $X$ have an upper fuzzy point. If $I_0^X$ is compact Hausdorff, then $X$ is compact $FT_{2w}$.

**Theorem 4.9.** Let $X$ be a $(q, \in)-FT_1$-space. If $I_0^X$ is locally compact at each $E \in K(X)$, then $X$ is locally F-compact.

**Theorem 4.10.** Let $X$ be a normalized $FT_{2w}$-space. If $X$ is locally F-compact, then $K(X)$ is open in $I_0^X$.

**Theorem 4.11.** Let $X$ be a $(q, \in)-FT_1$ and $FT_{2w}$-space and let $x_\lambda \in F_p(X)$. If $I_0^X$ is locally compact at $\{x_\lambda\}$, then $K(X)$ is locally compact at $\{x_\lambda\}$.

**Theorem 4.12.** Let $X$ be a $(q, \in)-FT_1$ and $FT_{2w}$-space and let $x_\lambda \in F_p(X)$. If $K(X)$ is locally compact at $\{x_\lambda\}$, then $X$ is locally F-compact at $x_\lambda$.

From Theorem 4.11 and Theorem 4.12, we can easily obtain the following.

**Corollary 4.12.** Let $X$ be a $(q, \in)-FT_1$ and $FT_{2w}$-space and let $x_\lambda \in F_p(X)$. If $I_0^X$ is locally compact at $\{x_\lambda\}$, then $X$ is locally F-compact at $x_\lambda$.

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