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하이브리드 수의 조건부 기대값

Conditional Expectation of Hybrid Number

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Abstract

We propose some properties of fuzzy conditional expectation of hybrid number the addition of fuzzy number and random variable using Cartesian product distance for α-level sets.

Keywords : hybrid number, conditional expectation, fuzzy random variable, Cartesian product, sum-product convolution.

1. Introduction

We propose some properties of fuzzy conditional expectation of hybrid number the addition of fuzzy number and random variable.

Y. Feng[1] was defined the concepts of a Gaussian fuzzy random variable and Gaussian fuzzy random vector and discussed their properties. On random fuzzy variables of second order and their application to linear statistical inference with fuzzy data was introduced by A. Wunsche and W. Nather[9]. The properties conditional expectation of Gaussian fuzzy variable was suggested by A. Wunsche and W. Nather[7]. In their paper, let (Ω, A, P) be a probability space, $\tilde{X}, \tilde{Y}$ are two convex fuzzy random variables on R^n. They proved that conditional expectation $E(\tilde{X}|\tilde{Y})$ is the best approximation of $\tilde{Y}$ by measurable function of $\tilde{X}$.

Let

$$\tilde{Y} = E(\tilde{Y}) \oplus \{\xi_1\} \text{ and } \tilde{X} = E(\tilde{X}) \oplus \{\xi_2\} \quad (1.1)$$

where $E(\tilde{Y})$ and $E(\tilde{X})$ are Aumann expectation of $\tilde{Y}$ and $\tilde{X}$ and ξ_1, ξ_2 are classical Gaussian random vectors with

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right). \quad (1.2)$$

Then the conditional expectation results in

$$E(\tilde{Y}|\tilde{X}) = E\tilde{Y} \oplus \Sigma_{12}\Sigma_{22}^{-1}\xi_2, \quad (1.3)$$

i.e. $E(\tilde{Y}|\tilde{X})$ is also Gaussian fuzzy random variable. Further, linearity of $E(\tilde{Y}|\xi_2)$ leads to

$$E(\tilde{Y}|\xi_2) = E(E\tilde{Y} \oplus \{\xi_1\}|\xi_2)$$
$$= E\,\tilde{Y} \oplus E(\xi_1|\xi_2) = E\,\tilde{Y} \oplus \Sigma_{12}\Sigma_{22}^{-1}\xi_2 \quad (1.3)$$

for Hukuhara difference with

$$\{\xi_2\} = \tilde{X} \ominus_H E\tilde{X}. \quad (1.4)$$

We think that this not fuzzy conditioned expectation. Now, we want to represent that $E(\tilde{Y}|\tilde{X})$ are fuzzy conditioned expectation.

2. Some concept of hybrid number

We begin our analysis by reviewing the addition of fuzzy number and random variables. In the referential set R, we have a fuzzy number A with membership function $\chi_A(x)$ and a random variable L whose probability is given by the density function $f_L(x)$, where x is a value of L in R. We want to add A and L.

Suppose that $f_L(x)$ has a convex shape. Let the maximum value of $f_L(x)$ be

$$\max_x f_L(x). \quad (2.1)$$

Divide the function $f_L(x)$ by its maximum value and define the new function $\chi_A(x)$, which is both convex and normal. By this process we substitute the fuzzy number L for the random variable L, which allows L to be added to any other fuzzy number by the operation of maximum convolution.

We therefore have

$$\chi_L(x) = \frac{f_L(x)}{\max_x f_L(x)}, \quad (2.2)$$

and addition with a fuzzy number A,

$\forall x, y, z \in R$

$$\chi_{A \oplus L}(z) = \bigvee_{z=x+y} (\chi_A(x) \wedge \chi_L(y')). \quad (2.3)$$

Let us consider a fuzzy number $A \subset R$ for a move to the right, $l > 0$, or a move to left, $l < 0$, we can write $\forall \alpha \in [0,1]$:

$$A \oplus L = [a_1^{(\alpha)}, a_2^{(\alpha)}] \oplus [l, l]$$
$$= [a_1^{(\alpha)} + l,\ a_2^{(\alpha)} + l]. \quad (2.4)$$

Let a not be a specific ordinary number but a random variable L with a probability density $f(l)$. The fuzzy number A will move randomly according to the law $f(l)$.

The couplet (A, L) called a hybrid number. A hybrid number can be represented as

$$X_h = A(\chi, f) \text{ or } X_h = A \oplus L,$$

where L is the random variable with probability density $f(l)$. The probability density of $(A_\alpha \oplus l)$ is the density of $L = l$ as

$$g(A_\alpha \oplus l) = g([a_1^{(\alpha)} + l,\ a_2^{(\alpha)} + l]) = f(l). \quad (2.5)$$

3. Statistics of hybrid number

Consider two hybrid numbers in R :

(A_1, L_1) and (A_2, L_2), where L_1 and L_2 have the densities $f_1(l_1)$ and $f_2(l_2)$, respectively. We now define addition by the hybrid convolution

$$(A_1, L_1) \oplus (A_2, L_2)$$
$$= (A_1 \oplus A_2, L_1 \oplus' L_2) = (A, L) \quad (3.1)$$

where $\oplus$ represents the max-min convolution for addition and $\oplus'$ represents the sum-product convolution of addition.

We may also write

$\forall x, y, z \in R$

$$\chi_{A_1 \oplus A_2}(Z) = \bigvee_{Z=x+y} (\chi_{A_1}(x) \wedge \chi_{A_2}(x)) \quad (3.2)$$

and

$$f(l) = \int_R f_1(l - l_2) f_2(l_2) dl_2 = \int_R f_1(l_1) f_2(l - l_1) dl_1 \quad (3.3)$$

For a each closed interval R, $[a_1^{(\alpha)}, a_2^{(\alpha)}]$, We have

$$[\phi(a_1^{(\alpha)}), \phi(a_2^{(\alpha)})] \subset R \quad (3.4)$$

and for $l \in R$

$$[\phi(a_1^{(\alpha)} + l), \phi(a_2^{(\alpha)} + l)] \subset R. \quad (3.5)$$

Also, we have that (A, L) is convex, compact and normal.

If l is a value of the random variable L the lower bound and upper bounds of (3.5) depend only on l for a given level α.

Now, we can compute the mathematical expectation ;

$$E[\phi(a_1^{(\alpha)}+l),\phi(a_2^{(\alpha)}+l)] \quad (3.6)$$
$$=[\int_{-\infty}^{\infty}\phi(a_1^{(\alpha)}+l)f(l)dl,\int_{-\infty}^{\infty}\phi(a_2^{(\alpha)}+l)f(l)dl]$$

Theorem 3.1. The membership function of the mathematical expectation of hybrid number (A,L) is the membership function of A shifted by the mathematical expectation of L.

<proof>

Let us use the intervals of confidence for level α,

$$E_\alpha(A\oplus L)$$
$$=[\int_{-\infty}^{\infty}(a_1^{(\alpha)}+l)f(l)dl,\int_{-\infty}^{\infty}(a_2^{(\alpha)}+l)f(l)dl]$$
$$=[a_1^{(\alpha)}\int_{-\infty}^{\infty}f(l)dl+\int_{-\infty}^{\infty}lf(l)dl,$$
$$a_2^{(\alpha)}\int_{-\infty}^{\infty}f(l)dl+\int_{-\infty}^{\infty}lf(l)dl]$$
$$=[a_1^{(\alpha)}+E(L),a_2^{(\alpha)}+E(L)]. \quad (3.7)$$

4. Some distance of fuzzy number

We denote by fuzzy number in ε_N^n

$$A=(a_1,a_2,\cdots,a_n) \quad (4.1)$$

where $a_i(i=1,\cdots,n)$ are projection of A to axis $X_i(i=1,\cdots,n)$, fuzzy number in R, respectively.

Definition 4.1. The $\alpha-$level set of fuzzy number in ε_N^n is define by

$$[A]^\alpha=\{(x_1,\cdots,x_n)\in R^n: (x_1,\cdots,x_n)\in\prod_{i=1}^n[a_i]^\alpha\} \quad (4.2)$$

where notation $\prod$ is the Cartesian product of sets.

Definition 4.2. Let A and B in ε_N^n, for all $\alpha\in(0,1]$,

$$A=B\Leftrightarrow[A]^\alpha=[B]^\alpha \quad (4.3)$$
$$[A*_nB]^\alpha=\prod_{i=0}^n[a_i*b_i]^\alpha \quad (4.4)$$

where $*_n$ is operation in ε_N^n and $*$ is operation in ε_N.

Let $\prod_{i=0}^n[a_i]^\alpha$, $0<\alpha\leq1$, be a given family of nonempty areas.

If

$$\prod_{i=0}^n[a_i]^\beta\subset\prod_{i=0}^n[a_i]^\alpha \text{ for } 0<\alpha<\beta<1 \quad (4.5)$$

and

$$\prod_{i=0}^n\lim_{k\to\infty}[a_i]^{\alpha_k}=\prod_{i=0}^n[a_i]^\alpha \quad (4.6)$$

then the family

$\prod_{i=0}^n[a_i]^\alpha$, $0<\alpha\leq1$ represents the $\alpha-$level sets of a fuzzy number $A\in\varepsilon_N^n$, where $(\alpha_k)$is a nondecreasing sequence converging to $\alpha\in(0,1]$.

Conversely, if $\prod_{i=0}^n[a_i]^\alpha$, $0<\alpha\leq1$ are the $\alpha-$level sets of a fuzzy number in R^n, then conditions (4.5) and (4.6) are true.

We define the metric d_∞ on ε_N^n.

Definition 4.3. Let $A,B\in\varepsilon_N^n$,

$$d_\infty(A,B)=sup\{d_H([A]^\alpha,[B]^\alpha):\alpha\in(0,1]\}$$
$$=sup\left\{d_H(\prod_{i=1}^n[a_i^\alpha],\prod_{i=1}^n[b_i]^\alpha):\alpha\in(0,1]\right\}$$
$$=sup\left\{\sqrt{\sum_{i=1}^n d_H([a_i^\alpha],[b_i]^\alpha))^2}:\alpha\in(0,1]\right\} \quad (4.7)$$

where d_H is Hausdorff distance.

5. Fuzzy conditional expection

If we have Gaussian random variable and Gaussian fuzzy random variable addition to hybrid number as

$$X_h=X\oplus\tilde{X},\quad Y_h=Y\oplus\tilde{Y}. \quad (5.1)$$

From the Theorem 3.1, we have Aumann

expectation of X_h , Y_h with

$$(EY_h)_\alpha = E(Y_h)_\alpha, \qquad (EX_h)_\alpha = E(X_h)_\alpha, \qquad \forall \alpha \in [0,1] \tag{5.2}$$

If there exists $E\|Y_h\| < \infty$, $E\|X_h\| < \infty$ then we have variance

$$VarY_h = Ed_\infty(Y_h, EY_h) \ , \tag{5.3}$$

$$VarX_h = Ed_\infty(X_h, EX_h) \ . \tag{5.4}$$

The (5.3), (5.4) can be written in a well known form as

$$Var(Y_h) = E\langle Y_h, Y_h\rangle - \langle EY_h, EY_h\rangle, \tag{5.5}$$

$$Var(X_h) = E\langle X_h, X_h\rangle - \langle EX_h, EX_h\rangle. \tag{5.6}$$

where $\langle \cdot , \cdot \rangle$ is inner product.

Consequently we have,

$$Cov(X_h, Y_h) \stackrel{def}{=} E\langle X_h, Y_h\rangle - \langle EX_h, EY_h\rangle$$

and (5.7)

$$\rho(X_h, Y_h) \stackrel{def}{=} \frac{Cov(X_h, Y_h)}{\sqrt{VarX_h}\sqrt{VarY_h}}. \tag{5.8}$$

For the properties of hybrid number, we have

$$\begin{pmatrix} Y_h \\ X_h \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{Y_h} \\ \mu_{X_h} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right) \tag{5.9}$$

where $\Sigma_{11} = VarY_h$, $\Sigma_{12} = Cov(Y_h, X_h)$

$\Sigma_{21} = Cov(X_h, Y_h)$ $\Sigma_{22} = VarX_h$.

Then we have following conditional expectation

$$E(Y_h|X_h) = EY_h + \Sigma_{12}\Sigma_{22}^{-1}(X_h - EX_h). \tag{5.10}$$

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