Robertson-walker 시공간의 지표정리 응용

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Application of Index form on the Robertson-Walker spacetime

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요 약

우주를 표현하는 시공간은 중력, 척력등 여러 가지 요인으로 심하게 굴곡되어진 공간일것이라는 이론이 관찰로 입증되었다. 그러므로 시공간의 연구는 다중 비틀림 공간의 연구를 있게 했고 앞으로도 더 많은 연구가 있게 될것으로 기대된다.
본 연구는 다중 비틀림 공간에서의 지표정리의 적용으로 Roberson-Walker 시공간에 대한 연구이다. 지표정리는 시공간의 특이성과 밀접한 관계를 가지므로 Roberson-Walker 시공간에서의 특이성조건을 조사하였다.

Abstract

We have the index form of the multiply warped products manifold. From this result we will investigate the physical properties on the Robertson-Walker spacetime. The singularty thorem are usually interpreted as an indication that black hole exist and that there has been a big bang. This paper include necessary condition for the Robertson-Walker spacetime with singularty.

1. Introduction

The dynamics of the expanding universe only appeared implicitly in the time dependence of the scale factor f(t). To make this time dependence explicit, one must solve for the evolution of the scale factor using the *Einstein field equations*:

$$R_{\mu\nu} - \frac{1}{2} S \; g_{\mu\nu} + \Lambda \; g_{\mu\nu} = 8\pi G \; T_{\mu\nu},$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter.

To describe the expansion of the universe one must use the Robertson-Walker metric along with Einstein equations. We may assume the entire universe is homogeneous and isotropic. The metric for a space with homogeneous and isotropic spatial sections is the *Robertson-Walker metric*, which can be written in the form

$$ds^2 = -dt^2 + f^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2\right),$$

where (t, r, θ, ψ) are coordinates, f(t) is the cosmic scale factor and k can be chosen to be -1, 0 or +1 for space of constant positive, negative or zero spatial curvature, respectively.

This can express the following statement.

Let B = (a,b) be a subset of R and let F be a connected three-dimensional Riemannian manifold of constant curvature k. Let f be a positive smooth function on B.

Then M = B $\times_f$ F is a Lorentzian warped product with the line element

$$g = -dt^2 + f^2\, d\Omega^2,$$

where $d\Omega^2$ is the line element of F.

When k = -1, 0 or 1, The Lorentizan warped product manifold M = B $\times_f$ F is called a *Robertson-Walker spacetime.*

2. Standard model: Robertson-Walker spacetime.

For X, Y and Z tangent to all slices F(t) and flow vector field U= ∂_t on M, [8]

(1) $R_{XY}Z = [\left(\frac{f}{f}\right)^2 + \frac{k}{f^2}](g(X,Z)Y - g(Y,Z)X).$

(2) $R_{XU}U = \left(\frac{f'}{f}\right)X.$

(3) $R_{XY}U = 0.$

(4) $R_{XU}Y = \left(\frac{f'}{f}\right)g(X,Y)U.$

(5) $Ric(U,U) = -3\left(\frac{f'}{f}\right).$

(6) $Ric(U,X) = 0.$

(7) $Ric(X,Y) = [2\left(\frac{f}{f}\right)^2 + \frac{k}{f^2} + \frac{f'}{f}]g(X,Y).$

And the scalar curvature

(8) $S = 6[\left(\frac{f}{f}\right)^2 + \frac{k}{f^2} + \frac{f'}{f}].$

2.1 The Robertson-Walker flow.

For interval (0 $\in$)I, the collection of all integral curve $I_\alpha \rightarrow M$ with $\alpha(0)$=p and $\alpha'(0)$=V, all these curve define a single integral curve $\alpha_p : I_p \rightarrow M$ where $I_p = \bigcup I_\alpha$. We call α_p the maximal integral curve of V starting at p. For V $\in \Xi(M)$ let q = $\alpha_p(s)$.

Then s + I_p = I_q and $\alpha_p(s+t) = \alpha_q(t)$ for all t $\in I_q$. Every maximal integral curve is either one-to-one, simply periodic or constant.

A vector field is *complete* if each of its integral curve is defined on the entire real line.

The *flow* of a complete vector field V on M is the map ψ : M $\times$R $\rightarrow$ M given by

$$\psi(p,t) = \alpha_p(t),$$

where α_p is the maximal integral curve starting at p. Hence the flow consists of one-parameter group of transformations.

A curve $\alpha(s)$ = (t(s), $\beta(s)$) in a Robertson-Walker spacetime (I $\times_f$ F, $-dt^2 + f^2(t)\, g_F$) is a geodesic if and only if [8]

(1) $\frac{d^2t}{ds^2} + g_F(\beta', \beta')f(t)\frac{df(t)}{dt} = 0,$

(2) $\beta'' + 2\frac{1}{f(t)}\frac{df(t)}{dt}\frac{dt}{ds}\beta' = 0.$

2.2 Singularity.

Since the stress-energy tensor T of a Robertson-Walker spacetime is already determined by the Einstein field equations, we can find the energy density function ρ and the pressure function p.

We will consider Λ = 0.

Theorem 2.1. [7] If U is the flow vector field on a Robertson-Walker spacetime, then

$$\frac{8\pi\rho}{3} = \left(\frac{f}{f}\right)^2 + \frac{k}{f^2},$$

$$-8\pi p = \frac{2f'}{f} + \left(\frac{f}{f}\right)^2 + \frac{k}{f^2}.$$

This takes the following formula.

$$\frac{3f'}{f} = -4\pi(\rho + 3p).$$

Also, the force equation is trivial and the energy equation is

$$\rho' = -3(\rho + p)\frac{f}{f}.$$

Now, we will consider $\Lambda \neq 0$.

The Einstein field equations and the energy conservation law can be written

(1) $8\pi G\rho + \Lambda = 3H^2 + 3K$,

(2) $8\pi Gp - \Lambda = -2H' - 3H^2 - K$,

where $H = \frac{f}{f}$ is the Hubble parameter and $K = \frac{k}{f^2}$.

The equation of state is written as

$$p = (\omega - 1)\rho$$

where γ is a constant. One may then use (1) and (2) to find the useful formula

$$3\left(\frac{f'}{f} + \Lambda\right) = -4\pi(3\omega - 2)\rho.$$

Proporsition 2.2

(1) For $\Lambda < 0$, if $p < \frac{-\rho}{3}$, then f has a minimal point.

(2) For $\Lambda > 0$, if $p > \frac{-\rho}{3}$, then f has a maximal point.

The equations

$$\frac{R''}{R} = \left(1 - \frac{3\omega}{2}\right)\left[\left(\frac{f}{f}\right)^2 + \frac{k}{f^2}\right] + \frac{\gamma\Lambda}{2},$$

$H = \sqrt{\frac{(3\omega-2)K - \omega\Lambda}{(2-3\omega)+2q}}$ obtains the following Proposition.

Proporsition 2.3. For the γ, if there is t_* such that $\Lambda(t_*) \neq \frac{(3\omega-2)K}{\omega}$ and $q(t_*) = \frac{2-3\omega}{2}$, then R has a singularity at $t = t_*$

We will classify the flat universes.

On taking $\omega = 0$ ($p = -\rho$) the following relation is

$$\Lambda = -8\pi G\rho.$$

On taking $\omega = \frac{2}{3}$ the following relation is

$$\Lambda = \frac{-8\pi G\rho(1+q)}{2+q}.$$

On taking $\gamma = 1$ ($p = 0$) the following relation is

$$\Lambda = \frac{4\pi G\rho(1-2q)}{1+q}.$$

On taking $\omega = \frac{3}{4}$ (radiation dominated universe) the following relation is

$$\Lambda = \frac{8\pi G\rho(1-q)}{1+q}.$$

Next, we will consider $\Lambda(t)$.

(1) $$f'' = \frac{-4\pi G\rho f}{3} + \frac{12\pi G'\rho f^2}{f} + \frac{\Lambda f}{3} + \frac{\Lambda' f^2}{6f},$$

(2) $\Lambda' = -24\pi GH(\rho + p) - 8\pi G' - 8\pi G\rho'$,

where H is the Hubble parameter.

Here, consider the static universe. From above (1) and (2)

$$\Lambda = 4\pi G\rho - 12\omega\pi G\rho + 4\pi(G'\rho + G\rho)\frac{1}{H} - 16\pi G'\rho f.$$

On taking $\Lambda = 0$ and $\omega = 0$, ($\rho = -p$), the following relation become that G is a constant and $4\pi G\rho = 0$.

3. Application of Index form.

For the geodesic curve γ, the index form $I : V^{\perp}(\gamma) \times V^{\perp}(\gamma) \rightarrow R$ is defined the symmetric bilinear form given by

$$I(X,Y) = -\int_a^b \frac{1}{\sqrt{\|\alpha'\|_{I^2} - f^2\|\beta'\|_{F^2}}} [-g(Y, R(X,\gamma')\gamma') + g(X',Y') + g(\gamma', A) + \frac{f\partial^2 f}{\partial v^2}|_{v=0} \|\beta'\|^2] du.$$

In the Robertson-Walker spacetime (k=0) with metric the timelike geodesic in the plane $\theta = \frac{\pi}{2}$ are straight lines when r and ϕ are interpreted as the usual polar coordinates. Then the geodesics are given by

$$r = \int_{t_0}^{t} \frac{c}{f\sqrt{f^2 + c^2}} dt,$$

where t_0 and c is an arbitrary constant. i.e., the geodesic

$$\gamma = (t, \int_{t_0}^{t} \frac{c}{f\sqrt{f^2 + c^2}} dt, \frac{\pi}{2}, \text{constant}).$$

Example 3.1. Let f be defined by $f(u) = u$, $u \in [1, \infty)$, and let $(u,v) \rightarrow u\cosh v \rightarrow f(u\cosh v)$, $v \in (-\delta, \delta)$. Let $X = aU + bY = (X^t, X^r, X^\theta, X^\phi) \in V^{\perp}(\gamma)$ be a timelike vector field and $\gamma' = U + Z$. Since $g(X,X) = -a^2 + b^2$ $g(Y,Y) < 0$ and $g(Y,Y) < \frac{a^2}{b^2}$.

Then $I(X,X) < 0$. Hence X is incomplete. This means that the Robertson-Walker spacetime (k=0) has a singularty.

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