Near Subtraction Semigroups에 관한 연구

연 용 호, 김 미 숙, 김 미 혜
충북 대학교

On Near Subtraction Semigroups

Yon Yong-Ho, Kim Mi-Suk, Kim Mi-Hye
Chungbuk National University

요 약

B.M. Schen([2])은 함수의 합성 " ∘ "과 차집합 연산 " − "에 대하여 닫혀있는 함수들의 집합 Φ 에서의 대수적 구조인 subtraction semigroup $(\Phi; \circ, -)$를 정의하였다. 이 구조에서 $(\Phi; \circ)$ 는 semigroup, $(\Phi; -)$는 [1]에서 정의한 subtraction algebra를 이룬다. B.M. Schen은 [2]에서 모든 subtraction semigroup은 invertible function들의 difference semigroup과 동형이라는 사실을 밝혔다.
본 논문에서는 이 subtraction semigroup의 한 일반화로써 near subtraction semigroup을 정의하고 이의 한 특수한 형태인 strong near subtraction semigroup의 개념을 정의하여 이들의 일반적인 성질과 ideal의 특성을 조사하고 이들의 응용도를 조사하고자 한다.

Abstract

B. M. Schein([1]) considered systems of the form $(\Phi; \circ, -)$, where Φ is a set of functions closed under the composition " ∘ " of functions and the set theoretic subtraction " − ". In this structure, $(\Phi; \circ)$ is a function semigroup and $(\Phi; -)$ is a subtraction algebra in the sense of [1]. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Also this structure is closely related to the mathematical logic, Boolean algebra, Bck-algera, etc.

In this paper, we define the near subtraction semigroup as a generalization of the subtraction semigroup, and define the notions of strong for it, and then we will search the general properties of this structure, the properties of ideals, and the application of it.

1. Introduction

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "-" that satisfies the following identities: for any $x, y, z \in X$,

(1) $x-(y-x)=x$;

(2) $x-(x-y)=y-(y-x)$;

(3) $(x-y)-z=(x-z)-y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$.

If the subtraction algebra X has the zero element 0 such that $0=a-a$ for all $a \in X$, then the subtraction determines an order relation on X:

$$a \leq b \Leftrightarrow a-b=0.$$

We will consider the subtraction algebra that has the element 0.

The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is , it is a meet semilattice with 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here

$$a \wedge b = a-(a-b),$$

and the complement b' of an element $b \in [0, a]$ is $a-b$, and for any $b, c \in [0, a]$,

$$\begin{aligned} b \vee c &= (b' \wedge c')' \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

The subtraction algebra X has the following properties ([4]):

(1) $x-0=x$ and $0-x=0$.

(2) $x-(x-y) \leq y$.

(3) $x \leq y \;\Leftrightarrow\; x=y-w$ for some $w \in X$.

(4) $x \leq y$ implies $x-z \leq y-z$ for all $z \in X$.

(5) $x \leq y$ implies $z-y \leq z-x$ for all $z \in X$.

(6) $x-(x-(x-y))=x-y$.

By a *subtraction semigroup*([3]) we mean an algebra $(X;\cdot,-)$ with two binary operations "$-$" and "$\cdot$" that satisfies the following axioms:

(1) $(X;\cdot)$ is a semigroup,

(2) $(X;-)$ is a subtraction algebra,

(3) for any $x, y, z \in X$, $x(y-z)=xy-xz$ and $(x-y)z=xz-yz$.

2. Definition of near subtraction semigroup

Definition 2.1. By a *near subtraction semigroup* (shortly, denoted by NSS) we mean an algebra $(S;\cdot,-)$ with two binary operations "$-$" and "$\cdot$" that satisfies the following axioms:

(1) $(S;\cdot)$ is a semigroup,

(2) $(S;-)$ is a subtraction algebra,

(3) $a(b-c)=ab-ac$ for any $a, b, c \in S$..

It is a *left* near subtraction semigroup in strictly, and if S satisfies the following:

$$(3'):(a-b)c=ac-bc$$

then it is called a *right* near subtraction semigroup, and if S is a left and right near subtraction semigroup, then S is a subtraction semigroup.

Example 2.2. Let $S=\{0,1\}$ and "$-$" and "$\cdot$" are defined by

$-$	0	1
0	0	0
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Then S is a NSS.

Example 2.3. Let $S=\{0,1,2,3,4,5\}$ and "$-$" and "$\cdot$" are defined by

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

Then S is a NSS.

Example 2.4. In example 2.3, the operations " − " and " · " are defined by

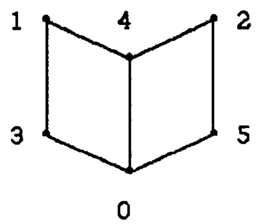
−	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

·	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	1	2	3	4	5
2	0	1	2	3	4	5
3	0	1	2	3	4	5
4	0	1	2	3	4	5
5	0	1	2	3	4	5

Then S is a NSS.

In example 2.3, S is commutative, that is, $ab=ba$ for all $a, b\in S$, , but in example 2.4, S is not commutative.

We can draw the Hasse diagram for S in example 2.3 and 2.4 as follow:



Example 2.5. Let Z_+ be the set of all positive integer and $S(Z_+)=\{\overline{n}:n\in Z_+\}$, where $\overline{n}=\{nz:z\in Z_+\}$ for each $n\in Z_+$. We can define two binary operations " − " as the set theoretic subtraction and " · " as the following:

$$\overline{n}\cdot\overline{m}=\overline{n}\cap\overline{m}.$$

for every $\overline{n}, \overline{m}\in S$. Then S is a NSS, and generally, $\overline{n}\cdot\overline{m}\supseteq\overline{nm}$. If n and m are relatively prime, i.e., $(n, m)=1$, then

$$\overline{n}\cdot\overline{m}=\overline{nm}.$$

Theorem 2.6.([4]) If S is a NSS, then $(S;\leq)$ is a poset with the relation given as:

$$a\leq b \text{ if and only if } a-b=0$$

for any $a, b\in S$.

Theorem 2.7.([4]) If S is a NSS, then $a\wedge b$ is the greatest lower bound of a and b with $a\wedge b=a-(a-b)$ for any $a, b\in S$.

A NSS S is a lower semilattice with the order $\leq$ from Theorem 2.7, and 0 is the bottom element in S.

Lemma 2.8. If S is a NSS, then

(1) $a0=0$ for all $a\in S$,

(2) If $a\leq b$, then $ca\leq cb$, for all $a, b, c\in S$.

(Proof) (1) For any $a\in S$,

$$a0=a(0-0)=a0-a0=0.$$

(2) Let $a\leq b(a, b\in S)$. Then $a-b=0$, and we have

$$ca-cb=c(a-b)=c0=0,$$

hence $ca\leq cb$.

3. Ideal in near subtraction semigroups

Definition 3.1. Let $(S;-,\cdot)$ be a NSS and I a nonempty subset of S. Then I is called a *left ideal* (resp. *right ideal*) in S if

(1) I is a subalgebra of $(S;-)$,

(2) $SI\subseteq I$ (resp. $IS\subseteq I$).

I is called an *ideal* in S if I is both a left and right ideal in S.

Example 3.2. In Example 2.3 and 2.4, $I=\{1, 1, 3, 4\}$ is an ideal in S.

Lemma 3.3. If S is a NSS, then for any $a\in S$, $aS=\{as:s\in S\}$ is a right ideal in S.

(Proof) If as_1, $as_2 \in aS$, then

$$as_1 - as_2 = a(s_1 - s_2) \in aS,$$

hence aS is a subalgebra of the subtraction algebra $(S; -)$, and since $(aS)S = a(SS) \subseteq aS$, aS is a right ideal in S.

As shown in example 2.4, generally, $0a \neq 0$ and $Sa = \{sa : s \in S\}$ is not a left ideal of S.

If a NSS S has a element 1 such that

$$a1 = 1a = a$$

for all $a \in S$, then the element 1 is called a *unity* in S.

Definition 3.4. A NSS S is *strong*(denoted by SNSS) if for all $a, b \in S$,

$$a - b = a - a \cdot b.$$

If S is a SNSS with the unity 1, then 1 is the greatest element in S, since

$$x - 1 = x - x1 = x - x = 0.$$

Example 3.5. Let $X = \{0, a, b, 1\}$ in which two binary operation " $-$ " and " $\cdot$ " are defined as follows

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then S is a SNSS with the unity 1.

Lemma 3.6. Let S be a SNSS. Then

(1) $ab \leq b$ for any $a, b \in S$,

(2) $a \leq b$ if and only if $a \leq ab$, for any $a, b \in S$.

(Proof) (1) Let $a, b \in S$. Then we have

$$ab - b = ab - (ab)b = ab - a(bb)$$
$$= a(b - bb) = a(b - b)$$
$$= a0 = 0,$$

hence $ab \leq b$.

(2) It is easy to show from the definition of SNSS and the above (1).

Corollary 3.7. If S is a SNSS, then for any $a \in S$, a is an idempotent element.

Theorem 3.8. Let S be a SNSS with 1. Then the following are equivalent :

(1) I is a right ideal in S.

(2) $b \in I$ and $a \leq b$ imply $a \in I$.

(Proof) Let I be a right ideal in S. If $b \in I$ and $a \leq b$, then $a = b - w$ for some $w \in S$, and we have

$$a = b - w = b - bw \in I,$$

since $bw \in IS \subseteq I$.

Conversely, suppose that $b \in I$ and $a \leq b$ imply $a \in I$. Let $a, b \in I$, Then since $1 - b \leq 1$,

$$a - b = a - ab = a1 - ab = a(1 - b)$$
$$\leq a1 = a \in I,$$

i.e., $a - b \in I$ and I is a subalgebra of S. Also if $a \in I$ and $s \in S$, then

$$as - a = as - a1 = a(s - 1) = a \cdot 0 = 0,$$

and hence $as \leq a \in I$, and $as \in I$.

Theorem 3.9. If S is a SNSS with 1, then

$$a \wedge b = ab$$

for all $a, b \in S$.

Corollary 3.10. If S is a SNSS with 1, then

$$aa = a$$

for all $a \in S$ and S is a commutative, i.e., S is an idempotent and commutative subtraction semigroup.

Corollary 3.11. If S is a SNSS with 1, then the following are equivalent :

(1) I is an ideal in S.

(2) $b \in I$ and $a \leq b$ imply $a \in I$.

(Proof) It is proved immediately from Theorem 3.8 and commutativity of S.

참 고 문 헌

[1] J. C. Abbott, "Sets, Lattices and Boolean Algebras", Allyn and Bacon, Boston, 1969.

[2] B. M. Schein, "Difference Semigroups, Comm. in Algebra" Vol. 20, pp. 2153-2169, 1992.

[3] B. Zelinka, "Subtraction Semigroups", Math. Bohemica, Vol. 120, pp. 445-447, 1995.

[4] K. H. Kim, E. H. Roh, Y. H. Yon, "On Subtraction Semigroups", Semigroup Forum, Submitted.