Deriving Probability Models for Stress Analysis

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Abstract

This paper presents an approach to derive probability models for use in structural reliability studies. Two main points are made. First, that it is possible to translate engineering and physics knowledge into a requirement on the form of a probability model. And second, that making assumptions about a probability model for structural failure implies either explicit or hidden assumptions about material and structural properties. The work is foundational in nature, but is developed with explicit examples taken from planar and general stress problems, the von Mises failure criterion, and a modified Weibull distribution.

1. Introduction

An important design parameter for structural analysis is the probability that a structure or component will fail. This number is generally calculated from material constants, the probable stresses to be applied to the component, a failure criterion (such as the von Mises or Tresca criteria), experimentation, and the selection of a probability model (such as the normal, exponential or Weibull distributions). Although there has been much discussion on the relative merits of the various failure criteria, and much experimental work in describing material constants such as Young's modulus and the Poisson ratio, relatively little work has been done to direct the selection of an appropriate probability model, or to connect model parameters to physically measurable values.

Probability models are often selected by choosing one of the standard analytical models, such as the normal, exponential or Weibull models, and fitting the parameters to experimental data. This paper discusses an alternate approach to selecting a probability model. It is a method of *deriving* models that are consistent with physical laws and established engineering judgement. Similarly, the approach illustrates a technique for determining the physical assumptions that are implied by assuming a particular probability model. The work is foundational in nature, but

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applications to failure analysis for isotropic materials are given. Because of this, the results are relevant to a wide variety of structural engineering problems.

2. Problem Definition

Frequently, a critical point (CP) of failure is identified in a component, and the probability of failure $p(fail)$ for the component is taken to be the probability of failure at the CP. Assume for the time being, that the CP has been identified, and that information regarding the probable states of stress S is available. The probability that the material will fail can then be found by applying the law of total probability

$$p(fail) = \int p(fail \mid S)\, f(S)\, dS,$$

where $p(S) \overset{\text{def}}{=} f(S)dS$ is the probability of being in a given stress state S, with probability density $f(S)$. The problem is to understand how the components of this generic probability model relate to the underlying question of material failure. The present work restricts its attention to yielding under static loading. We intend to extend the techniques. however, to dynamic loading conditions in future work.

First consider the probability model $p(S)$ for the stresses S which are applied to the CP. One factor that is important in determining a successful model is historical or design data. Two other factors which are equally important are one, the physical design and symmetries of the material near the CP, and two, the choice of coordinates for expressing the state of stress. For instance, many failure criteria use information about S in terms of the principal stresses (σ_1, σ_2, σ_3)[1] On the other hand, sensors may report information regarding the observed strains or stresses (σ_x, $\tau_{xy}, \cdots$) In addition, there may be information regarding the orientation of the principal stresses, which may be expressed relative to the Euler angles $(\phi, \theta, \psi)$of the principal stress directions. A successful probability model for the state of stress $p(S)$ will take into consideration each of these data requirements. There are difficulties in constructing such a model, however.

1) The maximum stress criteria and von Mises criteria are most simply expressed in terms of 3 coordinates-the principal stresses, even though more complicated forms which are expressed in terms of 6 coordinates- (σ_x, $\tau_{xy}, \cdots$)

A general state of stress for an isotropic material is 6 dimensional quantity. Therefore, the probability model should have six degrees of freedom. But more than 6 coordinates have been identified as relevant to the problem of stress, including the principal stresses, measurements, and principal stress directions. Section 3 describes how to encapsulate the information available about each of these parameters into a single coherent probability model by discussing stress states of linear elastic isotropic materials in 2 and 3 dimensions.

Consider next the probability of material failure under a particular state of stress, $p(fail \mid S)$. This probability is often taken to be 0 or 1, depending on whether the selected failure criterion predicts failure. For instance, with the von Mises criterion, failure is predicted with probability 1 if the distortional strain energy exceeds a given parameter Θ If Θ is not known, however, $p(fail \mid S)$ may be any value [0,1] Lindquist[1] derives a Weibull-type model for Θ and in Section 4 we discuss and extend the result by indicating what assumptions on the material are implied by assuming different parameter values in the Weibull model.

3. Probability Models for an Observed Stress

In this section, we examine by way of two examples, the translation of engineering statements about material stresses into probability models. In the first example, planar stresses are analyzed, and the second example treats stresses in 3-dimensional materials. Both examples make a similar assumption, namely, it is assumed that the principal stresses are equally likely to come from any direction. In the plane, this means that the principal stress coordinate frame is rotated uniformly from 0 to π radians. A similar statement holds for 3-dimensional materials.

In both cases, the goal is to find a probability model which is parametrized in terms of some observation (such as σ_x, τ_{xz}) and the principal stresses. The model must satisfy the additional assumption regarding the randomness of the principal stress directions. Such a model can then be used to update the probability of failure, given that certain measurements have been made. For instance, it may be of value to calculate

$$p(fail \mid \sigma_x) = \int p(fail \mid S, \sigma_x) f(S \mid \sigma_x) dS.$$

Since σ_x is information which is contained in the stress tenser S, the formula can be slightly simplified as shown in the equation

$$p(fail \mid \sigma_x) = \int p(fail \mid S) f(S \mid \sigma_x) dS.$$

3.1 Planar stress states

A planar stress tensor has three degrees of freedom, $(\sigma_x, \sigma_y, \tau_{xy})$. Equivalently, these degrees of freedom may be specified by the principal stresses and a rotation angle specifying their direction $(\sigma_1, \sigma_2, \phi)$ This notation is summarized on a sketch of the familiar Mohr circle(Figure 3.1).

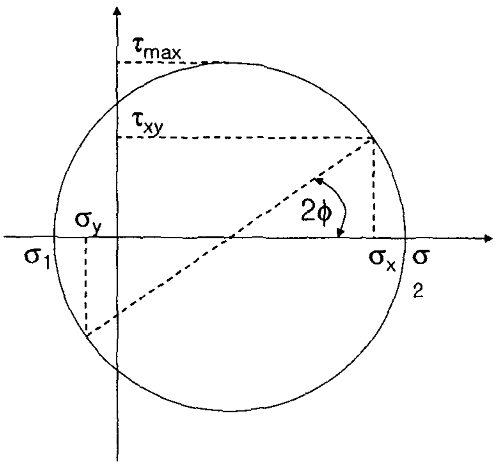

Figure 3.1 A Mohr circle for an arbitrary planar stress.

The statement that ϕ is randomly selected is the same as saying that ϕ is uniformly distributed on $(0, \pi)$. It also implies that the rotation angle ϕ is independent of the principal stresses. If the joint probability of the principal stresses is given by $f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2$ the probability of observing a given state of stress is then

$$\begin{aligned} p(S) &= p(\sigma_1, \sigma_2, \phi) \\ &= p(\sigma_1, \sigma_2 \mid \phi) p(\phi) \\ &= p(\sigma_1, \sigma_2) p(\phi) \\ &= \frac{f(\sigma_1, \sigma_2)}{\pi} d\sigma_1 d\sigma_2 d\phi. \end{aligned}$$

Measurements of a planar state of stress are typically made in terms of τ_{xy}, σ_x, or σ_y rather than the rotation angle. Assume the observation τ_{xy} has been made. To make predictions about the principal stresses given this information, rewrite $p(S)$ in terms of the coordinates $(\sigma_1, \sigma_2, \tau_{xy})$. This can be done by the Jacobian rule for probability densities and the formulas

$$\tau_{xy} = (\sigma_2 - \sigma_1)\cos\phi\sin\phi$$
$$d\sigma_1 d\sigma_2 d\tau_{xy} = (\sigma_2 - \sigma_1)\cos(2\phi)\, d\sigma_1 d\sigma_2 d\phi.$$

Since $\sin(2\phi) = \tau_{xy}/|\sigma_1 - \sigma|/2$ the probability model can be written

$$p(S) = \frac{f(\sigma_1, \sigma_2)}{(\sigma_2 - \sigma_1)\pi}\left(1 - \frac{\tau_{xy}^2}{\left(\frac{\sigma_2 - \sigma_1}{2}\right)^2}\right)^{-1/2} d\sigma_1 d\sigma_2 d\tau_{xy} \qquad (1)$$

where the stress tensor S is expressed in terms of $(\sigma_1, \sigma_2, \tau_{xy})$.

Although there is a great deal of freedom in modeling the strength of the principal stresses with an arbitrary $f(\sigma_1, \sigma_2)d\sigma_1 d\sigma_2$ the rest of the probability model is determined by the assumption that the principal stress directions are completely random. Any model which can be written in the form of Equation (1) is consistent with this assumption, and any model which can not be expressed in this form is *not* consistent with the assumption. In this sense, we have derived Equation (1) based on the physical assumptions of the problem. This probability model does not normally appear in introductory probability texts, but it has been described before. It is a special case of the l_2-isotropic probability model discussed in Mendel[2].

The same process produces a similar result if σ_x is the given data, rather than τ_{xy}. In Equation (1), replace τ_{xy} with $\sigma_x - |\sigma_1 + \sigma_2|/2$ and $d\tau_{xy}$ with $d\sigma_x$. This result is easily seen relative to the Mohr circle, since τ_{xy} and $\sigma_x - |\sigma_1 + \sigma_2|/2$ are coordinates for the 2-d state of stress, whose distance is a constant from the center of the Mohr circle.

The above models break down when the principal stresses are the same, $\sigma_1 = \sigma_2$ This special case, however, represents a pure hydrostatic stress, and is handled in a straightforward manner by most criteria for failure.

3.2 General stress states

A more complicated problem is that of general stress in a solid, isotropic component. Here, there are six degrees of freedom, often specified by the observable stress tensor components $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ Alternately, the stress tensor can be specified by the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ and their directions. Since these directions are orthogonal, they can be specified by the Euler angles (ϕ, θ, ψ) in 3 dimensions. In this section, we derive a probability model which is consistent with the notion that the 3 principal stress directions may be oriented randomly. This is the same as saying that the set of all stress tensors with the same principal stresses should somehow be 'uniformly' distributed. The model will be expressed in terms of the principal stresses and three observable stress components to facilitate inference on the probability of failure once the value of some, but not all, stress components are known.

We analyze first what is meant by a random orientation. The Euler angle triple, (ϕ, θ, ψ) represents a rotation of $\phi \in (-\pi, \pi)$ radians about the z-axis, followed by a rotation of $\theta \in (0, \pi)$ about the new x-axis, and then a rotation of $\psi \in (-\pi, \pi)$ about the new z-axis. Each Euler angle rotation can be thought of as an orientation for the principal stress directions, and it can be shown (Miles[3]) that a random orientation is given by

$$p(\phi, \theta, \psi) = \frac{1}{8\pi^2} \sin\theta \, d\phi \, d\theta \, d\psi,$$

where the constant normalizes the probability to 1.

Therefore, a complete probability model is given by

$$p(S) = \frac{f(\sigma_1, \sigma_2, \sigma_3)}{8\pi^2} \sin\theta \, d\theta_1 d\theta_2 d\theta_3 d\phi \, d\theta \, d\psi,$$

where $f(\sigma_1, \sigma_2, \sigma_3)$ is any probability density for the principal stresses. Again, proceed using the Jacobian rule to replace the angle measurements with observables from the stress matrix. This gives the following conditional probability model for $\sigma_z, \tau_{xz}, \tau_{yz}$ conditioned on knowing the values of the principal stresses:

$$p(\sigma_z, \tau_{xz}, \tau_{yz} \mid \sigma_1, \sigma_2, \sigma_3) = \frac{d\sigma_z d\tau_{xz} d\tau_{yz}}{8\pi^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)\sin 2\phi \, \sin^2\phi}.$$

The complete model is therefore

$$p(S) = \frac{f(\sigma_1, \sigma_2, \sigma_3)\, d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_z d\tau_{xz} d\tau_{yz}}{8\pi^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)\sin 2\phi\ \sin^2\phi}$$

$$= \frac{f(\sigma_1, \sigma_2, \sigma_3)\, d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_z d\tau_{xz} d\tau_{yz}}{16\pi^2(-\alpha_{12}\alpha_{23}\alpha_{31})^{1/2}},$$

where the stress tensor S is parametrized in terms of the principal stresses and three observables, σ_z, τ_{xz}, τ_{yz} and the α_{ij} are given by

$$\alpha_{ij} = (\sigma_i - \sigma_z)(\sigma_j - \sigma_z) + \tau_{xz}^2 + \tau_{yz}^2.$$

A derivation of this is due to Tsai[4].

Once again, this model gives the entire class of probability distributions which are consistent with the assumption that the principal stresses may come from any direction whatsoever with equal probability. It also enables the inference on the probability of structural failure by giving a conditional probability model for observables in terms of the principal stresses. Bayes rule can be applied to calculate the probability of failure given the observables. This is particularly helpful in a dynamic environment where preventative action may be taken if an undesirable situation arises.

Notice that the model presented here is not valid some of the principal stresses are the same, because a division by 0 would be required. This is handled by two cases. First, suppose that all three principal stresses were the same. This represents the case of pure hydrostatic stress, and no special probability model is required. If exactly two of the three principal stresses are the same, there are only 4 degrees of freedom to the stress tensor. Two of the degrees of freedom are $\sigma_1 = \sigma_2$ and σ_3, the other two being the principal stress direction for σ_3, which can be given by the first two Euler angles(ϕ, θ). The ψ is neglected because rotations in the σ_1, σ_2 stress plane result in the same stress tensor. The general model which satisfies the given assumptions is therefore

$$p(S \mid \sigma_1 = \sigma_2) = \frac{f(\sigma_1, \sigma_3)}{4\pi} \sin\theta\, d\sigma_1 d\sigma_3 d\phi d\theta.$$

4. Probability Models for Failure Given the Stress

Consider the problem of describing the probability of failure given that a certain state of stress is applied to it. Timoshenko[5] presents a number of criteria as a basis, including the maximum stress, maximum strain, maximum shearing stress(Tresca), and maximum distortion energy (von Mises-Hencky) criteria. At face value, these criteria predict failure if their parameter exceeds a certain critical level. For instance, with the von Mises criterion, material failure is predicted if the total distortion energy U_{dist} exceeds the mean distortion energy capacity θ of the material. This section discusses two different approaches to determine what θ should be, and describes the connection between the parameters of these models and hidden assumptions regarding the distribution of distortion energy in a material.

For the sake of preciseness, assume the problem is that of studying the failure of a ductile, linear, isotropic material with the von Mises criterion.[2)] In this case, the model predicts material failure if the total distortion energy exceeds the mean distortion energy capacity θ of a material.

If the mean distortion energy capacity θ of a material were known precisely, the von Mises model for failure $p(fail \mid S)$ could be modeled as

$$p(fail \mid S) = \begin{cases} 1 \text{ if } U_{dist} \geq \theta \\ 0 \text{ if } U_{dist} < \theta \end{cases}.$$

However, since variability in material strength exists due to the variability of material properties or potential flaws and cracks, θ cannot be specified by a fixed value. Further, there may be restrictions on the amount of uniaxial stress test data which is available for estimating θ.

A probability model can be used to describe this inherent uncertainty regarding θ. With such a model for θ, the probability of failure can be rewritten as

$$p(fail \mid S) = p(\theta \geq U_{dist}).$$

Weibull[6] proposed the following model which is applied to a wide variety of problems in engineering, including the determination of a probability density for yielding stress, and in turn for θ. The model gives the probability for component failure under a uniaxial stress x, and has 3 parameters as seen in the equation

2) Note that although the von Mises criterion is discussed in particular, a similar analysis for the other failure models can be made.

$$p(x \mid x_u, m, x_0) = \frac{m}{x_0}(x - x_u)^{m-1} \exp\left(-\frac{(x - x_u)^m}{x_0}\right) dx. \tag{2}$$

Here, x_u is the zero strength, measured in units of stress, m is a shape parameter for the model, and x_0 is correlated with θ. 3)

Although Equation (2) is largely motivated by engineering judgement, Weibull admits that the function $(x - x_u)^m / x_0$ was chosen in an ad hoc fashion. His criteria was to choose a function that disappears at the zero strength x_u, is easy to write, and is positive and non-decreasing. He did not see a way to derive, from engineering judgements, any of these 3 parameters.

Recently, Lindquist[1] proposed that the function in the exponent of Equation (2) can be derived from assumptions about how distortion energy is distributed in a material. In particular, he shows that the function $(x - x_u)^m / x_0$ is more reasonably written as $(x^2 - x_u^2)/(6G\theta)$ where G is the shearing modulus. This differs in two ways from Weibull's equation. First, the exponent applies to the individual terms rather than their difference, and second, the parameters are identified with physically meaningful quantities $-x_0$ is shown to be a function of the shearing modulus and the mean distortion energy capacity, and m is 2 because distortion energy is proportional to the square of the uniaxial stress $(u = \sigma^2/6G)$. Here, $x_u^2/(6G)$ is interpreted as the minimum distortion energy capacity of a material.

It is useful to tersely summarize Lindquist's derivation before making additional comments. First, assume that a batch of material is sufficiently large to make N stress test rods. If rod i breaks with distortion energy u_i due to the uniaxial stress σ_i, then the average distortion energy is

$$\theta = \frac{1}{N} \sum_{i=1}^{N} u_i = \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_i^2}{6G}.$$

From Mendel[2], the following likelihood is derived for the first n test rods, conditioned on the average density. Note that we maintain the assumption that a rod never fails with a distortion energy less than $e_u = \sigma_u^2/6G$.

3) The parameters are traditionally found by statistical parameter estimation of Bayesian inference based on Equation (2) and a set of uniaxial stress test data taken from experiment with sample rods.

$$p(u_1,\ldots,u_n \mid \theta) = c_N\left(1 - \frac{\sum_{i=1}^{n}(u_i - e_u)}{N\theta}\right)^{N-1} du_1 \cdots du_n.$$

The proportionality factor c_N normalizes the probability to 1. For very large batch sizes, and with a change of variables, this model can be approximated by the exponential function given in the equation

$$p(\sigma_1,\ldots,\sigma_n \mid \theta) = c \prod_{i=1}^{N} \frac{\sigma_i}{6G\theta - \sigma_u^2} \exp\left(-\frac{\sigma_i^2 - \sigma_u^2}{6G\theta - \sigma_u^2}\right) d\sigma_1 \cdots d\sigma_n. \quad (3)$$

This derivation explicitly points out that a slightly modified form of Weibull's distribution is required, given certain physical considerations and the assumption that average distortion energy alone is relevant to predicting failure.

Although many strong assumptions were made in this section, there are several interesting points which arise as results of the derivation of Equation (3). First, we reiterate that Weibull chose the function $(x-x_u)^m / x_0$somewhat arbitrarily, although it possesses many qualitative features which seem to match the problem of failure prediction. Next, Lindquist has shown how a slightly modified form of Weibull's function can be derived from the physical notion of average distortion energy. In this framework, parameters assume an operational meaning. In addition, we see that if the new model's analog of m is assumed to be something other than 2, the notion that average distortion energy in the samples is relevant to the failure criterion is violated. This may indicate that another quantity, such as the average applied stress when $m = 1$, is relevant, or that there are additional factors involved with the failure process. This seems to be the case for non-ductile, nonlinear, or non-isotropic materials, where standard statistical fitting techniques have given values for m over a wide range (say, from 2 to 26). Additional parameters, therefore, may be required to model the probability of failure in these cases. In any event, other probability models certainly make physical assumptions about the failure process, although they may be more difficult to mathematically describe.

5. Conclusions

Although the models developed in this paper made strong assumptions which do not apply to all structural reliability problems, several widely applicable conclusions may be drawn. One, it is possible to convert physical properties and engineering judgements into classes of probability models. It is not necessary to arbitrarily select them. In the case here, a statement about principal stress directions was used to derive a class of models. A key tool is the Jacobian rule for probability densities, which permits engineering statements to be preserved regardless of the coordinate system chosen to express them. Two, parameters in a probability or statistical model have meaning relative to the engineering assumptions that apply to a system. In particular, we saw how the Weibull parameters for a material strength model relate to the way strength is assumed to be distributed through a collection of components made from the same material. Much work remains to be done, however, in deriving other model types and determining the physical assumptions inherent in other probability models Three, it is unnecessary to base models on parameters which have an abstract and non-measurable definition. All of the parameters derived here have meaning relative to the problem of structural reliability. To be sure, for the sake of computational efficiency, it may be simpler to use such an abstract parameter. However, the use of such a parameter does make hidden assumptions about the physics of the problem.

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