The Busy Period of the $M/M/1$ Queue with Bounded Workload

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Abstract

In this paper, with martingale argument we derive the explicit formula for the Laplace transform of the busy period of $M/M/1$ queue with bounded workload which is also called finite dam. Much simpler derivation than appeared in former literature is provided.

Keywords: $M/M/1$ queue; busy period; workload; finite dam; martingale

1 Introduction

We, in this paper, consider the $M/M/1$ queue with uniformly bounded workload $V > 0$, where customers arrive according to Poisson process of rate $\nu > 0$, the requested service times of customers are i.i.d. with common distribution function, $1-(1/m)e^{-(1/m)x}, x > 0$, and the workload of the queue is uniformly bounded by V. See Cohen(1969) for detailed definition. With the terminology in dam theory, our model is the dam of finite capacity V in which the input process is formed by a compound Poisson process with arrival rate ν and i.i.d. exponential inputs whose means are m. The workload in the queue corresponds to reservoir in the dam and busy period to wet period. In order to present the Laplace transform of the busy period of the model, we use martingale method, which results in direct and much simpler derivation of the transform.

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In many case of studying queues or dams, complex analysis is applied. On the other hand, the method of Kolmogorov's backward differential equation is also used. For example, using the method, Kinateder and Lee(2000a) obtained the Laplace transform of the time to being idle or overflow of workload, whose consequence is the Laplace transform of busy period, in the model of $M/M/1$ queue with bounded workload. Recently, Kim et al. obtained the Laplace transform of busy period, even for general model, $M/G/1$ queue with bounded workload.

Another tool which can be used to study the workload processes of queues is a martingale method. Usually, it is probablistic and avoids tedious calculations. Using martingale argument, Rosenkrantz(1983) derived a formula of the Laplace transform of the busy period of $M/G/1$ queue. Lee and Kinateder(2000b) find the expected value of the busy period of $M/M/1$ queue with bounded workload. In Kinateder and Lee(2000a), a formula for the Laplace transform of the busy period of the same model is appeared, but they adopt not only martingale method but also the technique which Feller(1971) shows for computing Laplace transform for standard Brownian motion with absorption, and because of it the derivation is long and devious.

We present, in this paper, the direct derivation which is simple and easily understandable using only optional sampling theorem in martingale theory.

2 Laplace Transform of a Stopping Time

Let $A(t)$ denote the arrival process of customers in the queue and let $S_1, S_2, ...$ be the i.i.d. random variables which represent the amounts of the works incurred by the arriving customers. An argument similar to that of Rosenkrantz(1983) shows that if process $X(t)$ is defined by

$$X(t) = x + \sum_{i=1}^{A(t)} S_i - t$$

then

$$M(t) = \exp\left(-sX(t) - t\left(s + \frac{\nu}{1+ms} - \nu\right)\right)$$

is a martingale with respect to $X(t)$ for all $s > -1/m$.

We define a stopping time T_x by

$$T_x = \inf\{t | X(t) > V \text{ or } X(t) = 0\}.$$

Then, the Optional Stopping Theorem can be applied as like in Kinateder and Lee(2000a). Hence, we have

$$M(0) = E[M(T_x)],$$

that is,

$$\exp(-sx) = E\left[\exp\left(-sX(T_x) - \left(s + \frac{\nu}{1+ms} - \nu\right)T_x\right)\right].$$

Given that $X(T_x) > V$, $X(T_x)$ is the sum of V and the amount of overflow of which distribution is exponential with mean m. Therefore, the equation can be rewritten as

$$\begin{aligned}
&\exp(-sx) \\
&= \Pr\{X(T_x) > V\}\frac{\exp(-sV)}{1+ms}E\left[\exp\left(-\frac{ms^2-(\nu m-1)s}{1+ms}T_x\right)\,\middle|\, X(T_x) > V\right] \\
&\quad + \Pr\{X(T_x) = 0\}E\left[\exp\left(-\frac{ms^2-(\nu m-1)s}{1+ms}T_x\right)\,\middle|\, X(T_x) = 0\right], \qquad (1)
\end{aligned}$$

for $s > -1/m$.

Now, we let

$$\beta = \frac{ms^2-(\nu m-1)s}{1+ms} = s - \nu + \frac{\nu}{ms+1}.$$

Then s takes two values for each $\beta > 0$, and we say

$$\begin{aligned}
s_1(\beta) &= \frac{m\beta+\nu m-1+\sqrt{(m\beta+\nu m-1)^2+4m\beta}}{2m}, \\
s_2(\beta) &= \frac{m\beta+\nu m-1-\sqrt{(m\beta+\nu m-1)^2+4m\beta}}{2m}
\end{aligned}$$

Substituting $s_1(\beta)$ and $s_2(\beta)$ into equation (1) yields simultaneous equation on two restricted Laplace transforms of the stopping time $T(x)$.

$$\begin{aligned}
\frac{\exp(-s_1V)}{1+ms_1}A_x(\beta)+B_x(\beta) &= \exp(-s_1x), \\
\frac{\exp(-s_2V)}{1+ms_2}A_x(\beta)+B_x(\beta) &= \exp(-s_2x),
\end{aligned}$$

where $A_x(\beta) = E\left[\exp(-\beta T_x)\cdot 1_{\{X(T_x)>V\}}\right]$ and $B_x(\beta) = E\left[\exp(-\beta T_x)\cdot 1_{\{X(T_x)=0\}}\right]$.

Solving the above simultaneous equation, we get

$$\begin{aligned}
A_x(\beta) &= \frac{\exp(-s_1x)-\exp(-s_2x)}{\exp(-s_1V)/t_1-\exp(-s_2V)/t_2} \\
B_x(\beta) &= \frac{\exp(-s_1V-s_2x)/t_1-\exp(-s_2V-s_1x)/t_2}{\exp(-s_1V)/t_1-\exp(-s_2V)/t_2},
\end{aligned} \tag{2}$$

where $t_1 = t_1(\beta) = 1+ms_1(\beta)$ and $t_2 = t_2(\beta) = 1+ms_2(\beta)$.

3 Laplace Transform of Busy Period

Consider the workload process of the queue with initial workload x. Then the process is equal in distribution to process $X(t)$ until the process exceeds V or reaches 0. If we let τ_x be the busy period, we observe from the Markov property of $X(t)$ that

$$\tau_x \stackrel{\mathcal{D}}{=} T_x + \tau_V \cdot 1_{\{X(T_x)>V\}}.$$

Therefore, we have

$$\begin{aligned}
E[\exp(-\beta\tau_x)] &= \Pr\{X(T_x)>V\}E\left[\exp(-\beta T_x+\beta\tau_V)\,\middle|\, X(T_x)>V\right] \\
&\quad +\Pr\{X(T_x)=0\}E\left[\exp(-\beta T_x)\,\middle|\, X(T_x)=0\right] \\
&= A_x(\beta)E\left[\exp(-\beta\tau_V)\right]+B_x(\beta).
\end{aligned}$$

Putting $x=V$ in the equation gives

$$E\left[\exp(-\beta\tau_V)\right] = \frac{B_V(\beta)}{1-A_V(\beta)},$$

and hence

$$E[\exp(-\beta\tau_x)] = A_x(\beta)\frac{B_V(\beta)}{1-A_V(\beta)} + B_x(\beta).$$

Finally, we substitute $A_x(\beta)$ and $B_x(\beta)$ in equation (2) into the above equation to build the following theorem:

Theorem 1 *The Laplace transform of the busy period with initial workload x is given by*

$$E[\exp(-\beta\tau_x)] = \frac{(1/t_1-1)\exp(-s_1V-s_2x)-(1/t_2-1)\exp(-s_2V-s_1x)}{(1/t_1-1)\exp(-s_1V)-(1/t_2-1)\exp(-s_2V)},$$

$\beta > 0$.

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