Some results on weak laws of large numbers for weighted sums of fuzzy random variables

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Abstract

In this paper, we establish weak laws of large numbers for weighted sums of convex-compactly uniformly integrable fuzzy random variables taking values in the space of upper-semicontinuous fuzzy sets in R^p.

keywords: Fuzzy random variables, Weak law of large numbers, compactly uniform integrability, tightness, weighted sums

1 Introduction

The theory of fuzzy sets introduced by Zadeh (1965) has been extensively studied and applied in statistics and probability areas in recent years. Puri and Ralescu (1986) introduced the concept of fuzzy random variables as a natural generalization of random sets. Statistical inference for fuzzy random variables led to the need for laws of large numbers in order to ensure consistency in estimation problems.

In this paper, we give some results on weak laws of large numbers for weighted sums of fuzzy random variables taking values in the space of upper-semicontinuous fuzzy sets in R^p.

2 Preliminaries

Let $K(R^p)$ denote the family of non-empty convex and compact subsets of the Euclidean space R^p. Then the space $K(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

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A norm of $A \in K(R^p)$ is defined by

$$\|A\| = h(A, \{0\}) = \sup_{a \in A} |a|.$$

It is well-known that $K(R^p)$ is complete and separable with respect to the Hausdorff metric h. The addition and scalar multiplication on $K(R^p)$ are defined as usual:

$$\begin{aligned} A \oplus B &= \{a + b : a \in A, b \in B\} \\ \lambda A &= \{\lambda a : a \in A\} \end{aligned}$$

for $A, B \in K(R^p)$ and $\lambda \in R$.

Let $F(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0, 1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;

(2) $\tilde{u}$ is upper semicontinuous;

(3) $\tilde{u}$ is fuzzy convex, i.e. $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R^p$ and $\lambda \in [0, 1]$.

(4) $supp\,\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > 0\}$ is compact.

For a fuzzy set $\tilde{u}$ in R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that for each $\alpha \in [0, 1]$,

$$\tilde{u} \in F(R^p) \text{ if and only if } L_\alpha \tilde{u} \in K(R^p).$$

The linear structure on $F(R^p)$ is defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0, \end{cases}$$

for $\tilde{u}, \tilde{v} \in F(R^p)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

Now, we define the metric d_∞ on $F(R^p)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \sup_{x \in L_0 \tilde{u}} |x|.$$

Then it is well-known that $F(R^p)$ is complete but is not separable with respect to d_∞(see Klement et al. (1986)). Joo and Kim (2000) introduced a new metric d_s on $F(R^p)$ which makes it a separable metric space as follows:

Definition 2.1 *Let T denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in F(R^p)$, we define*

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 \ : \ \text{there exists a } t \in T \text{ such that} \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

3 Main Results

Throughout this paper, let (Ω, A, P) be a probability space. A set-valued function $X : \Omega \to K(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B) = \{\omega : X(\omega) \cap B \neq \emptyset\} \in A.$$

It is well-known that the measurability of X is equivalent to the measurability of X considered as a map from Ω to the metric space $(K(R^p), h)$ A set-valued function $X : \Omega \to K(R^p)$ is called a random set if it is measurable.

A random set X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(\xi) : \xi \in L(\Omega, R^p) \,\text{and}\, \xi(\omega) \in X(\omega) \, a.s.\},$$

where $L(\Omega, R^p)$ denotes the class of all R^p-valued random variables ξ such that $E|\xi| < \infty$.

A fuzzy set valued function $\tilde{X} : \Omega \to F(R^p)$ is called a fuzzy random variable if for each $\alpha \in [0,1], L_\alpha \tilde{X}$ is a random set. Recently, Kim (2002) showed that a fuzzy random variable can be identified with a random element of the metric space $(F(R^p), d_s)$.

Thus, throughout the remainder of this paper, we assume that the space $F(R^p)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded fuzzy random variable $\tilde{X}$ is a fuzzy subset $E(\tilde{X})$ of R^p defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0,1] : x \in E(L_\alpha \tilde{X})\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrably bounded fuzzy random variable let $\{\lambda_{ni}\}$ be double array of nonnegative real numbers such that

$$\sum_{i=1}^{\infty} \lambda_{ni} \leq C$$

for all n and for some constant C.

In this section, the following three statements will be considered;

(a) $d_s(\oplus_{i=1}^{n} \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^{n} \lambda_{ni}E(\tilde{X}_i)) \to 0$ in probability.

(b) For each $\alpha \in [0,1]$,

$$h(\oplus_{i=1}^{n} \lambda_{ni}L_\alpha\tilde{X}_i, \oplus_{i=1}^{n} \lambda_{ni}E(L_\alpha\tilde{X}_i)) \to 0 \ \text{ in probability.}$$

(c) For each dyadic rational $\alpha \in [0,1]$,

$$h(\oplus_{i=1}^{n} \lambda_{ni}L_\alpha\tilde{X}_i, \oplus_{i=1}^{n} \lambda_{ni}E(L_\alpha\tilde{X}_i)) \to 0 \ \text{ in probability.}$$

It is trivial that (a) implies (b) and (b) implies (c). Thus, we will consider the condition to guarantee that (c) implies (a). To this end, we need the following concepts.

Definition 3.1 *Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.*

(1) $\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$ there exists a compact subset A of $F(R^p)$ such that

$$P(\tilde{X}_n \notin A) < \epsilon \text{ for all } n.$$

If A is convex and compact, then $\{\tilde{X}_n\}$ is said to be convexly tight.

(2) $\{\tilde{X}_n\}$ is said to be compactly uniformly integrable if for each $\epsilon > 0$ there exists a compact subset A of $F(R^p)$ such that

$$\int_{\{\tilde{X}_n \notin A\}} \|\tilde{X}_n\| \, dP < \epsilon \text{ for all } n.$$

If A is convex and compact, then $\{\tilde{X}_n\}$ is said to be convex-compactly uniformly integrable.

If $\{\tilde{X}_n\}$ is convexly tight, then it is tight. But the converse is not true. Similar statement for compactly uniform integrability holds.

Our main results are as follows;

Theorem 3.2 *Let $\{\tilde{X}_n\}$ be a sequence of convex-compactly uniformly integrable fuzzy random variables. Then (c) implies (a).*

Corollary 3.3 *Let $\{\tilde{X}_n\}$ be a sequence of level-wise independent and convex-compactly uniformly integrable fuzzy random variables. Then the assertion (a) is true.*

Corollary 3.4 *Let $\{\tilde{X}_n\}$ be a sequence of convexly tight fuzzy random variables such that*

$$\sup_n E\|\tilde{X}_n\|^r = M < \infty \text{ for some } r > 1.$$

Then (c) implies (a).

The r-th moments condition in Corollary 3.3 can not be replaced by the 1-st moments condition. But by requiring identical distributions, we can obtain the similar results.

Theorem 3.5 *Let $\{\tilde{X}_n\}$ be a sequence of convexly tight and identically distributed fuzzy random variables such that*

$$E\|\tilde{X}_1\| < \infty.$$

Then (c) implies (a).

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