2002 Proceedings of the Autumn Conference, Korean Statistical Society

Complete convergence for weighted sums of AANA random variables

TAE-SUNG KIM[1] AND MI-HWA KO[2]

Abstract

We study maximal second moment inequality and derive complete convergence for weighted sums of asymptotically almost negatively associated(AANA) random variables by applying this inequality.
2000 Mathematics Subject Classification : 60F05

Keywords : Complete convergence; Weighted sum; Negatively associated; Asymptotically almost negatively associated.

1. Introduction

Recall that a finite family $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$, $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$ (see Joag-Dev and Proschan(1983)). Matula(1992) has established a maximal inequality for negatively associated(NA) sequences. By inspecting the proof of Matula's(1992) maximal inequality, Chandra and Ghosal(1996) found that one can also allow positive correlations provided they are small. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} &Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\ &\quad \leq q(m)(var(f(X_m))var(g(X_{m+1}, \cdots, X_{m+k})))^{1/2} \end{aligned} \tag{1}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right side of (1) is finite.
In this paper we study complete convergence for weighted sums of AANA sequence, which has never been studied previously in the literature.

[1]Professor, Department of Mathematics, WonKwang University Iksan, Jeonbuk 570-749
E-mail : starkim@wonkwang.ac.kr

[2]Instructor, Department of Mathematics, WonKwang University Iksan, Jeonbuk 570-749
E-mail : songhack@wonkwang.ac.kr

2. Results

Lemma 2.1 Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated(AANA). Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables, where $f_n(\cdot), n = 1, 2, \cdots$, are nondecreasing functions.

Lemma 2.2 (Chandra and Ghosal (1996)) Let $X_1, \cdots, X_n$ be mean zero, square integrable random variables such that (1) holds for $1 \leq m < k + m \leq n$ and for all coordinatewise nondecreasing continuous functions f and g whenever the right-hand side of (1) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $1 \leq k \leq n$. Then we have

$$E(\max_{1\leq k\leq n} \sum_{i=1}^{k} X_i)^2 \leq (A + (1 + A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{2}$$

Theorem 2.3 Let $\{X_k, k \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$ *for all* $k \geq 1$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying the condition

$$\textstyle\sum_{k=1}^{n} a_{nk}^2 = O(n^\delta) \quad as \quad n \to \infty \; for \quad some \quad 0 < \delta < 1. \tag{3}$$

Assume that

$$\sum_{m=1}^{\infty} q^2(m) < \infty. \tag{4}$$

Then, $\forall\, \epsilon > 0$ and δ' such that $\delta < \delta' \leq 1$

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_{ni} X_i |> \epsilon n^{1/2}) < \infty. \tag{5}$$

Proof . To prove (5) it suffices to show that

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_{ni}^{+} X_i |> \epsilon n^{1/2}) < \infty \qquad \forall\, \epsilon > 0, \tag{6}$$

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_{ni}^{-} X_i |> \epsilon n^{1/2}) < \infty \qquad \forall\, \epsilon > 0, \tag{7}$$

where $a_{ni}^{+} = a_{ni} \vee 0$, $a_{ni}^{-} = (-a_{ni}) \vee 0$. We need only to prove (6), since the proof of (7) is analogous. From Lemma 2.1 $\{a_{ni}^{+} X_i, 1 \leq i \leq n, n \geq 1\}$ is an AANA sequence, and hence by applying we have Lemma 2.2

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\le k\le n} |\sum_{i=1}^{n} a_{ni}^{+} X_i |> \epsilon n^{1/2})$$
$$\le \epsilon^{-2} \sum_{n=1}^{\infty} n^{-1-\delta'} (A+(1+A^2)^{1/2})^2 \sum_{k=1}^{n} a_{nk}^2 EX_k^2 =: I.$$

Note that conditions (3) and $EX_k^2 < \infty$ imply

$$\sum_{k=1}^{n} a_{nk}^2 EX_k^2 = O(n^{\delta}) \quad as \quad n \to \infty. \tag{8}$$

Hence, by (4) and (8) we have

$$I << (A+(1+A^2)^{1/2})^2 \sum_{n=1}^{\infty} n^{-(1+\delta'-\delta)} < \infty \quad for \quad 0 < \delta < \delta' \le 1,$$

where $a << b$ means $a = O(b)$. The proof is completed.

Theorem 2.4 Let $\{X, X_k, k \ge 1\}$ be sequence of identically distributed AANA random variables with $EX = 0,\ EX^2 < \infty$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and let $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of real numbers satisfying (3). If (4) and (5) hold then for some δ' such that $\delta < \delta' \le 1$

$$\sum_{n=1}^{\infty} n^{-\delta'} \sum_{j=1}^{n} P(| a_{nj} X_j |> n^{1/2}) < \infty. \tag{9}$$

Proof. From (5) we obviously get

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\le j\le n} | a_{nj} X_j |> n^{1/2}) < \infty\ , \tag{10}$$

$$P(\max_{1\le j\le n} | a_{nj} X_j |> n^{1/2}) \to 0 \quad as \quad n \to \infty. \tag{11}$$

Note that

$$P(\max_{1\le j\le n} | a_{nj} X_j |> n^{1/2}) = \sum_{j=1}^{n} P(| a_{nj} X_j |> n^{1/2}, \max_{1\le i\le j-1} | a_{ni} X_i |\le n^{1/2}).$$

Hence, we deduce that

$$\sum_{j=1}^{n} P(| a_{nj} X_j |> n^{1/2}) = P(\max_{1\le j\le n} | a_{nj} X_j |> n^{1/2})$$
$$+ \sum_{j=1}^{n} P(| a_{nj} X_j |> n^{1/2}, \max_{1\le i\le j-1} | a_{ni} X_i |> n^{1/2}). \tag{12}$$

Also, we have

$$\sum_{j=1}^{n} P(|\, a_{nj}X_j \,|> n^{1/2},\ \max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2})$$
$$= \sum_{j=1}^{n}\{E[I(|\, a_{nj}X_j \,|> n^{1/2})I(\max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2})]$$
$$-EI(|\, a_{nj}X_j \,|> n^{1/2})EI(\max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2})\}$$
$$+\sum_{j=1}^{n}\{EI(|\, a_{nj}X_j \,|> n^{1/2})EI(\max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2})\}$$
$$\le E\sum_{j=1}^{n}[I(|\, a_{nj}X_j \,|> n^{1/2}) - P(|\, a_{nj}X_j \,|> n^{1/2})]I(\max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2})$$
$$+\sum_{j=1}^{n} P(|\, a_{nj}X_j \,|> n^{1/2})P(\max_{1\le i\le n} |\, a_{ni}X_i \,|> n^{1/2}) = II + III. \tag{13}$$

Define

$$Y_{nj} = \begin{cases} a_{nj}X_j, & if\ a_{nj} \ge 0\ , \\ -a_{nj}X_j, & if\ \ a_{nj} < 0\ . \end{cases}$$

Then $\{Y_{nj}, 1 \le j \le n, n \ge 1\}$ and $\{I(Y_{nj} > n^{1/2})\}$ are AANA by Lemma 2.1. By the Cauchy-Schwarz inequality and Lemma 2.2,

$$|\, II \,| \quad = \quad |\, E\sum_{j=1}^{n}[I(|\, a_{nj}X_j \,|> n^{1/2}) - P(|\, a_{nj}X_j \,|> n^{1/2})]$$
$$\times I(\max_{1\le i\le j-1} |\, a_{ni}X_i \,|> n^{1/2}) \,|$$
$$\le [E(\sum_{j=1}^{n} I(|\, a_{nj}X_j \,|> n^{1/2}) - EI(|\, a_{nj}X_j \,|> n^{1/2}))^2$$
$$\times E(I(\max_{1\le i\le n} |\, a_{ni}X_i \,|> n^{1/2}))^2]^{1/2}$$
$$= [Var(\sum_{j=1}^{n} I(|\, a_{nj}X_j \,|> n^{1/2}))P(\max_{1\le i\le n} |\, a_{ni}X_i \,|> n^{1/2})]^{1/2}$$
$$\le [2\{Var[\sum_{j=1}^{n} I(Y_{nj} > n^{1/2})] + Var[\sum_{j=1}^{n} I(Y_{nj} < -n^{1/2})]\}$$
$$\times P(\max_{1\le i\le n} |\, a_{ni}X_i \,|> n^{1/2})]^{1/2}$$
$$\le [8\{\sum_{j=1}^{n} P(Y_{nj} > n^{1/2}) + \sum_{j=1}^{n} P(Y_{nj} < -n^{1/2})\}$$
$$\times P(\max_{1\le i\le n} |\, a_{ni}X_i \,|> n^{1/2})(A + (1 + A^2)^{1/2})^2]^{1/2}$$

$$\leq \frac{1}{2}\sum_{j=1}^{n} P(|\, a_{nj}X \,|> n^{1/2}) + 4(A + (1 + A^2)^{1/2})^2 \times P(\max_{1\leq i\leq n} |\, a_{ni}X_i \,|> n^{1/2}) \tag{14}$$

by $\sqrt{ab} \leq \frac{a+b}{2}$. From (12)-(14) we have

$$\frac{1}{2}\sum_{j=1}^{n} P(|\, a_{nj}X \,|> n^{1/2}) \leq \{1 + 4(A + (1 + A^2)^{1/2})^2\} P(\max_{1\leq i\leq n} |\, a_{ni}X_i \,|> n^{1/2}) + \sum_{j=1}^{n} P(|\, a_{nj}X \,|> n^{1/2}) P(\max_{1\leq i\leq n} |\, a_{ni}X_i \,|> n^{1/2}) \tag{15}$$

and from (11) we get

$$\sum_{j=1}^{n} P(|\, a_{nj}X \,|> n^{1/2}) << P(\max_{1\leq i\leq n} |\, a_{ni}X_i \,|> n^{1/2}), \tag{16}$$

for sufficiently large n. Therefore, from (10) and (16) the desired result (9) follows.

References

[1] Chandra, T. K., Ghosal, S. (1996 a) *Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables*, Acta. Math. Hungar. **32** 327-336.

[2] Hsu, P. L. and Robbins, H. (1947) *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci **33** (2) 25-31.

[3] Joag-Dev, K. Proschan, F. (1983) *Negative association of random variables with applications*, Ann. Statist. **11** 286-295.

[4] Matula, P. (1992) *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Probab. Lett. **15** 209-213.