# Pricing Path-Dependent Equity-Indexed Annuities

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## Abstract

Sales of equity-indexed annuities (EIAs) have rapidly increased, but the growth rates in sales have recently shown signs of slowing down because the current volatile equity market increases the costs of guarantees in EIAs. New EIAs need to be designed that are similar to existing EIAs but have a cheaper guarantee and a higher participation rate. This paper proposes three types of EIAs with higher participation rates: up-and-in barrier EIA, annual reset EIA with up-and-in barriers, and partial-time lookbackEIA. It also presents a probability distribution and the method of Esscher transforms, with which explicit pricing formulas for these EIAs are derived.

Key Words: equity-indexed annuities, options, participation rate, guarantee, and pricing

## 1. Introduction

In the deferred annuities market, the portion of traditional fixed annuities in annual sales has declined from one half in 1994 to one quarter in 1999. This is in part due to economic changes in the financial market: relatively low interest rates and a bullish stock market which have led actuaries to design new types of annuities that link return to stock market performance. One of these products is equity-indexed annuities (EIAs). When a stock index, typically the S&P 500, goes up, EIAs provide policyholders with a rate of return connected to the index return. When the index goes down, EIAs provide policyholders with a minimum guaranteed return. Since the first offering in 1995, EIAs have gained popularity. Sales of EIAs in 1996, 1997, 1998, 1999 and 2000 were at $1.5, $3.0, $4.3, $5.1 and $5.4 billion, respectively.

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However, pricing EIAs is challenging problem due to the complex payoff structure of EIAs. Evaluating the guarantee embedded in an EIA is difficult and often requires advanced stochastic modeling techniques. Two key factors for pricing EIAs are participation rate and indexing method. The participation rate is the percentage of the index return to be credited. Thus, the insurance company credits to the EIA policy the greater of the index return times the participation rate and a minimum guaranteed return. An indexing method is the method that the insurer uses to calculate the index return to the policyholder based on the index values in the contract term. There are several indexing methods such as point-to-point, lookback, and annual reset.

Tiong (2000) has derived several closed-form formulas for the EIAs listed above. As pointed out in the paper, the growth rates in sales have recently shown signs of slowing down because the current volatile equity market increases the costs of guarantees in the EIAs and hence decreases the participation rates. Hence, new EIAs with higher participation rates need to be designed that are similar to the existing EIAs but have a cheaper guarantee.

To make EIAs more attractive to customers, I shall propose an up-and-in barrier EIA, an annual reset EIA with up-and-in barriers and a partial-time lookback EIA. In addition, I shall present explicit pricing formulas for these proposed EIAs by using the method of Esscher transforms.

## 2. Esscher Transforms and Probability Distribution

Let $S(t)$ denote the time-$t$ price of an equity index. Assume that the index is constructed with all dividends reinvested. Assume also that for $t \geq 0$,

$$S(t) = S(0)e^{X(t)}$$

where $\{X(t)\}$ is a Brownian motion with drift $\mu$ and diffusion coefficient $\sigma$, and $X(0) = 0$. Thus $X(t)$ has a normal distribution with mean $\mu t$ and variance $\sigma^2 t$.

Let us briefly summarize a special case of the method of Esscher transforms developed by Gerber and Shiu (1996). For a nonzero real number $h$, the moment generating function of $X(t)$, $\mathrm{E}[e^{hX(t)}]$, exists for all $t \geq 0$. The stochastic process

$$\{e^{hX(t)}\mathrm{E}[e^{hX(1)}]^{-t}\}$$

is a positive martingale used to define a new probability measure $Q$. In technical terms, the process is used to define the Radon-Nikodym derivative $\mathrm{d}Q/\mathrm{d}P$, where $P$ is the original probability measure. We call $Q$ the Esscher measure of parameter $h$.

For a random variable $Y$ that is a real-valued function of $\{X(t), 0 \le t \le T\}$, the expectation of $Y$ under the new probability measure $Q$ is calculated as

$$\mathrm{E}[Y \frac{e^{hX(T)}}{\mathrm{E}[e^{hX(1)}]^T}] \tag{2.1}$$

which will be denoted by $\mathrm{E}[Y; h]$. The risk-neutral Esscher measure is the Esscher measure of parameter $h = h^*$ with respect to which the process $\{e^{-rt}S(t)\}$ is a martingale. Thus

$$\mathrm{E}[e^{-rt}S(t); h^*] = S(0). \tag{2.2}$$

Therefore, $h^*$ is the solution of

$$\mu + h^*\sigma^2 = r - \sigma^2/2. \tag{2.3}$$

Note that the process $\{X(t)\}$ is a Brownian motion with drift $\mu + h\sigma^2$ and diffusion coefficient $\sigma$ under the Esscher measure of parameter $h$.

Next, let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of $\boldsymbol{Z}$ is

$$\Pr(Z_1 \le a, Z_2 \le b, Z_3 \le c) = \Phi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}). \tag{2.4}$$

For $0 \le s \le t$, let

$$M(s, t) = \max\{X(\tau), s \le \tau \le t\}. \tag{2.5}$$

It can be shown that for $0 < s < t \le T$, the joint distribution function of $M(s, t)$ and $X(T)$ is

$$\begin{aligned} &\Pr(M(s, t) \le m, X(T) \le x) \\ &= \Phi_3(\frac{x - \mu T}{\sigma\sqrt{T}}, \frac{m - \mu t}{\sigma\sqrt{t}}, \frac{m - \mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}) \\ &\quad - e^{\frac{2\mu}{\sigma^2}m}\Phi_3(\frac{x - 2m - \mu T}{\sigma\sqrt{T}}, \frac{-m - \mu t}{\sigma\sqrt{t}}, \frac{m + \mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}). \end{aligned} \tag{2.6}$$

## 3. Up-and-In Barrier EIA

To increase the participation rate, let us propose an up-and-in barrier EIA as an alternative to point-to-point EIAs. If the index rises above a barrier for the monitoring period, the return credited to the policy will be the greater of the index return times the participation rate and a minimum guaranteed return. Otherwise, the return credited to the policy will be the minimum guaranteed return. Thus, the up-and-in barrier EIA provides customers with a higher participation rate than point-to-point EIAs.

Let us take a close look at the payoff of the up-and-in barrier EIA. Assume that the minimum guaranteed return is $g$ for the contract term, the participation rate is $\alpha$, the barrier is $B$, and the monitoring period is from time $s$ to time $t$ ($0 < s < t \le T$). Let $u = \log[B/S(0)]$ and $k = \log(1 + g/\alpha)$. Then the payoff can be expressed as follows:

$$\begin{aligned} &S(0)[1 + \alpha(e^{X(T)} - 1)], \ \text{if } X(T) > k \text{ and } M(s, t) > u \\ &S(0)(1 + g), \ \text{otherwise.} \end{aligned} \tag{3.1}$$

By the *fundamental theorem of asset pricing*, the factorization formula (2.4), and the distribution function (2.6), the time-0 value of the payoff (3.1) is

$$\begin{aligned} &S(0)e^{-rT}\mathrm{E}[(\alpha e^{X(T)} - \alpha - g)\mathrm{I}(X(T) > k, M(s, t) > u) + (1 + g); h^*] \\ &= S(0)[\alpha P_1(r+\tfrac{1}{2}\sigma^2, T) - (\alpha + g)e^{-rT}P_1(r-\tfrac{1}{2}\sigma^2, T) + e^{-rT}(1+g)] =: V(S(0), T). \end{aligned} \tag{3.2}$$

Here, $\Phi(\cdot)$ denotes the standard normal distribution function and $P_1(\mu, T)$ is

$$\begin{aligned} &\Phi(-\frac{k-\mu T}{\sigma\sqrt{T}}) - \Phi_3(-\frac{k-\mu T}{\sigma\sqrt{T}}, \frac{u-\mu t}{\sigma\sqrt{t}}, \frac{u-\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}) \\ &+ e^{\frac{2\mu}{\sigma^2}u}\Phi_3(-\frac{k-2u-\mu T}{\sigma\sqrt{T}}, \frac{-u-\mu t}{\sigma\sqrt{t}}, \frac{u+\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}). \end{aligned} \tag{3.3}$$

## 4. Annual Reset EIA with Up-and-In Barriers

Let us consider an annual reset EIA with up-and-in barriers. In each period, this EIA will provide customers with the greater of the annual index return times the participation rate and a minimum guaranteed rate if the maximum index value for each monitoring period rises above a barrier. Otherwise, the EIA will credit to the policyholder the minimum guaranteed rate as annual return.

Consider the total return of each period. Note that the total return is defined as the return plus one. For $0 < s < t \le T/n$, let

$$M_i = \max\{X(\tau) - X(t_{i-1}),\ t_{i-1} + s \le \tau \le t_{i-1} + t\},\ i = 1, 2, \ldots, n \tag{4.1}$$

where $t_i - t_{i-1} = T/n$, $t_0 = 0$ and $t_n = T$. Assume that the minimum guaranteed rate is $g$ for each period from time $t_{i-1}$ to time $t_i$, the participation rate is $\alpha$ for each period, and $k = \log(1 + g/\alpha)$. The level of the barrier in each period is $S(t_{i-1})e^u$ for some constant $u$. Write $X_i := X(t_i) - X(t_{i-1})$. Then the total return of each period is as follows:

$$\begin{aligned} &1 + \alpha(e^{X_i} - 1), \ \text{if } X_i > k \text{ and } M_i > u \\ &1 + g, \ \text{otherwise.} \end{aligned} \tag{4.2}$$

Thus, the payoff at time $T$ of the annual reset EIA with the up-and-in barriers is

$$S(0)\prod_{i=1}^{n}\ [(\alpha e^{X_i} - \alpha - g)\mathrm{I}(X_i > k, M_i > u) + (1+g)]. \tag{4.3}$$

Because $(X_i, M_i)$ has the same distribution as $(X(T/n), M(s, t))$ and $\{(X_i, M_i), i = 1, 2, \ldots, n\}$ are independent under the Esscher measure of parameter $h^*$, it follows from the fundamental theorem of asset pricing that the time-0 value of the payoff (4.3) is

$$S(0)e^{-rT}\mathrm{E}[\prod_{i=1}^{n}\ [(\alpha e^{X_i} - \alpha - g)\mathrm{I}(X_i > k, M_i > u)\ + (1+g)];\ h^*]$$

$$= S(0)\prod_{i=1}^{n}\ \mathrm{e}^{-rT/n}\mathrm{E}[(\alpha e^{X_i} - \alpha - g)\mathrm{I}(X_i > k, M_i > u) + (1+g);\ h^*] = S(0)[V(1, T/n)]^n. \tag{4.4}$$

## 5. Partial-Time Lookback EIA

Next, we propose a partial-time lookback EIA. The index return of this EIA will be the same as that of the continuous lookback EIA of Tiong (2000) except that the maximum index value is attained during a partial life of the policy. Thus the partial-time lookback EIA provides its policyholders with the greater of this index return times the participation rate and the minimum guaranteed return. Assume that the participation rate is $\alpha$, the minimum guaranteed return is $g$, and $k = \log(1 + g/\alpha)$. The payoff is as follows:

$$\begin{aligned} &S(0)[\alpha(e^{M(t,T)} - 1) + 1], \text{ if } M(t, T) > k \\ &S(0)(g + 1), \text{ otherwise.} \end{aligned} \tag{5.1}$$

The time-0 value of the payoff (5.1) can be derived as follows:

$$S(0)e^{-rT}\{\alpha\ \mathrm{E}[e^{M(t,T)}\mathrm{I}(M(t, T) > k);\ h^*] - (\alpha+g)\Pr(M(t, T) > k;\ h^*) + (1+g)\}$$

$$= S(0)e^{-rT}\{\alpha I_1(r-\tfrac{1}{2}\sigma^2, 1) - (\alpha+g)P_2(r-\tfrac{1}{2}\sigma^2) + (1+g)\}. \tag{5.2}$$

Here, $P_2(\mu)$ denotes

$$1 - \Phi_2(\frac{k-\mu T}{\sigma\sqrt{T}}, \frac{k-\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}) + e^{\frac{2\mu}{\sigma^2}k}\ \Phi_2(\frac{-k-\mu T}{\sigma\sqrt{T}}, \frac{k+\mu t}{\sigma\sqrt{t}}; -\sqrt{\frac{t}{T}}), \tag{5.3}$$

where $\Phi_2(\cdot, \cdot\,; \rho)$ is the standard bivariate normal distribution function with correlation coefficient $\rho$. For $c + \xi \neq 0$ and $\xi = 2\mu/\sigma^2$, the expectation $I_1(\mu, c)$ denotes

$$\mathrm{E}[e^{cM(t,T)}\mathrm{I}(M(t, T) > k)]$$

$$= \frac{2c+\xi}{c+\xi}\, e^{c\mu T + \frac{1}{2}c^2\sigma^2 T}\ \Phi_2(\frac{-k+(\mu+c\sigma^2)T}{\sigma\sqrt{T}}, \frac{(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \sqrt{1-\frac{t}{T}})$$

$$+\frac{\xi}{c+\xi}e^{(c+\xi)k}\Phi_2(\frac{k+\mu t}{\sigma\sqrt{t}},-\frac{k+\mu T}{\sigma\sqrt{T}};-\sqrt{\frac{t}{T}})$$

$$+\frac{\xi}{c+\xi}e^{c\mu t+\frac{1}{2}c^2\sigma^2 t}\Phi(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}})\Phi(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}). \qquad (5.4)$$

Note that the formula (5.4) is a generalization of the expectation (D21) of Huang and Shiu (2001).

## 6. Conclusion

This paper has presented the joint distribution function of $M(s, t)$ and $X(T)$ necessary for pricing EIAs and proposed three new types of EIAs to make EIAs more attractive in the deferred annuities market. The proposed EIAs have higher participation rates than point-to-point, annual reset, and continuous lookback EIAs, respectively. In addition, explicit pricing formulas for the proposed EIAs are given. I hope that the proposed EIAs will help increase the popularity of EIAs and that the pricing formulas will be useful for actuaries in managing the proposed EIAs.

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