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A study on Robust Estimation of ARCH models.

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Abstract

In financial time series, the autoregressive conditional heteroscedastic (ARCH) models have been widely used for modeling conditional variances. In many cases, non-normality or heavy-tailed distributions of the data have influenced the estimation methods under normality assumption. To solve this problem, a robust function for the conditional variances of the errors is proposed and compared the relative efficiencies of the estimators with other conventional models.

1. Introduction

The autoregressive conditional heteroscedastic (ARCH) model was proposed by Engle (1982) and has been a very useful tool to model changing variances in financial time series. In real data, it is widely known that financial data have to have heavy tail distributions rather than normal distributions. One of the solutions for the problem is to introduce the robust method. Recently, Koenker and Zhao (1996) proposed a robust estimation for ARCH models using the quantile regression method. Also, Jiang et al. (2001) proposed L_1 -estimation of ARCH models. In this paper, we propose a Huber-type robust estimation and compare the relative efficiencies of the estimators with Engle-type and absolute-type ARCH models.

2. Robust Estimation of ARCH models.

Consider the following time series model
$Y_t = \sqrt{h_t} \bullet e_t$, where $\{e_t\}$ are i.i.d. random variables with zero mean and finite variance σ^2 and $h_t = \alpha_0 + \alpha_1 \bullet \epsilon_{t-1}^2$, $\quad \alpha_0 \geqq 0, \; 0 < \alpha_1 < 1$.
This model was originally proposed by Engle(1982). Usually Y_t means the rate of returns such as stock prices or interest rate in financial data. In this paper, we focus on the analysis of the proposed estimators by comparing relative efficiencies with other estimators based on simulation.

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2.1 Huber-type modeling for ARCH processes

Consider the Huber type conditional variance such as

$$h_t = \alpha_0 + \alpha_1 \bullet \rho_k(\epsilon_{t-1})$$

where

$$\rho_k(\epsilon_{t-1}) = \begin{cases} \epsilon_{t-1}^2, & |\epsilon_{t-1}| \leqq k \\ 2k|\epsilon_{t-1}| - k^2, & |\epsilon_{t-1}| \geqq k \end{cases} \tag{1}$$

and $\rho_k(\bullet)$ is continuously differentiable. We know that $\rho_k(\bullet)$ is a hybrid between (2) and (3),where (2) and (3) are as follows.

$$\epsilon_t = \sqrt{h_t} \bullet e_t$$

$$h_t = \alpha_0 + \alpha_1 \bullet \epsilon_{t-1}^2 \tag{2}$$

,and $h_t = \alpha_0 + \alpha_1 \bullet |\epsilon_{t-1}|$ (3)

Hereafter, we call (2) as the Engle's ARCH and (3) as the absolute ARCH.

3. Model efficiency

The simulation procedure to compare performance of the estimators are as follows. First, we need to get the weighted least squares(WLS) estimators of α_0 & α_1 by minimizing

$$Q(\underline{\alpha}) = \sum (\epsilon_t^2 - h_t)^2 / h_t^2 \tag{4}$$

Assume first k is known (k = 1.0 or 1.5),then $\tilde{h_t} = \tilde{\alpha_0} + \tilde{\alpha_1}\rho_k(\epsilon_{t-1})$ is approximated by

$$\begin{pmatrix} \tilde{\alpha_0} \\ \tilde{\alpha_1} \end{pmatrix} = (X'X)^{-1}X'\underline{Y}$$

where $$X = \begin{pmatrix} 1 & \rho_k(\epsilon_0) \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ 1 & \rho_k(\epsilon_{n-1}) \end{pmatrix}, \quad \underline{Y} = \begin{pmatrix} \epsilon_1^2 \\ \bullet \\ \bullet \\ \bullet \\ \epsilon_n^2 \end{pmatrix} \tag{5}$$

Note that $\begin{pmatrix} \tilde{\alpha_0} \\ \tilde{\alpha_1} \end{pmatrix}$ is consistent & asymptotically normal under some regularity conditions but is not efficient.

Next we calculate $\begin{pmatrix}\tilde{\alpha_0}\\ \tilde{\alpha_1}\end{pmatrix}$ by minimizing

$\hat{Q}(\alpha) = \sum (\epsilon_t^2 - h_t)^2/\tilde{h_t}^2$ and the WLS estimator is given by

$$\begin{pmatrix}\tilde{\alpha_0}\\ \tilde{\alpha_1}\end{pmatrix} = (X'W^{-1}X)^{-1}X'W^{-1}\underline{Y} \qquad (6)$$

where $W^{-1} = diag(\tilde{h_1}^{-2}, \cdot\cdot\cdot, \tilde{h_n}^{-2})$, $\tilde{h_t}$ is in (5)

Under regularity conditions, $\begin{pmatrix}\tilde{\alpha_0}\\ \tilde{\alpha_1}\end{pmatrix}$ is consistent and asymptotically normal.

For evaluating model efficiency, consider

$\epsilon_t^2 = \alpha_0 + \alpha_1\rho_k(\epsilon_{t-1}) + \eta_t$, with residual sum of squares such as

$$\hat{\hat{Q}}(\alpha) = \sum_{t=1}^{n}(\epsilon_t^2 - \hat{h_t})^2/\hat{h_t^2} \overset{say}{=} R_1 \qquad (7)$$

,where $\hat{h_t} = \hat{\alpha_0} + \hat{\alpha_1}\rho_k(\epsilon_{t-1})$ (($\hat{\alpha_0}$, $\hat{\alpha_1}$ are WLS)

For Engle's ARCH,

$$R_2 = \sum (\epsilon_t^2 - \hat{h_t})^2/\hat{h_t^2}$$

where $\hat{h_t} = \widehat{\alpha_{0,E}} + \widehat{\alpha_{1,E}}\ \epsilon_{t-1}^2 \overset{say}{=} R_2 \qquad (8)$

and , $\begin{pmatrix}\widetilde{\alpha_{0,E}}\\ \widetilde{\alpha_{1,E}}\end{pmatrix}$ is the WLS estimator from Engle's ARCH(1)

For absolute ARCH,

$\hat{h_t} = \widehat{\alpha_{0,A}} + \widehat{\alpha_{1,A}}\ |\epsilon_{t-1}| \overset{say}{=} R_3$

where $\begin{pmatrix}\widetilde{\alpha_{0,A}}\\ \widetilde{\alpha_{1,A}}\end{pmatrix}$ is the WLS from absolute ARCH(1).

3.1 Simulation model

Consider the simulation model such as

$\epsilon_t = \sqrt{h_t} \cdot e_t$

,where $h_t = \frac{2}{3} + \frac{1}{3}\rho_k(\epsilon_{t-1})$, $\quad k = 1.0$ and $1.5 \qquad (9)$

First we generate e_t from the standard normal, contaminate normal and double exponential distributions. Then we generate $\epsilon_1, \cdots, \epsilon_{200}$ to compute

R_1, R_2, R_3 and the ratios $\frac{R_2}{R_1}$, $\frac{R_3}{R_1}$ tare defined as the relative efficiencies in this simulation. In the following tables, SN(0,1) is the standard normal and CN(0,A) means the contaminated normal like 0.0*SN(0,1) +01.*N(0,A) and DE means the double exponential distribution..

Table 1. Relative efficiencies (α_0=0.33,α_1=0.2, k=1.0)

	Initial value : α_0=0.33 α_1=0.2	
Ratio / Distribution	R1/R2	R1/R3
SN(0,1)	0.9883187	0.9972303
CN(0,2)	0.9818778	0.9970747
CN(0,3)	0.9719380	1.0003642
CN(0,5)	0.9445798	1.0411480
CN(0,9)	0.9306110	0.9909174
DE(1)	0.9178369	1.024233

Table 2. Relative efficiencies (α_0=0.33,α_1=0.2, k=1.5)

	Initial value : α_0=0.33 α_1=0.2	
Ratio / Distribution	R1/R2	R1/R3
SN(0,1)	0.9919777	0.9974288
CN(0,2)	0.9832152	0.9975671
CN(0,3)	0.9756530	1.0006368
CN(0,5)	0.9632008	1.0162690
CN(0,9)	0.9266866	1.0545705
DE(1)	0.9225608	1.040417

Table 3. Relative efficiencies (α_0=0.33, α_1=0.2,k=2.0)

Distribution \ Ratio	Initial value : α_0=0.33 α_1=0.2	
	R1/R2	R1/R3
SN(0,1)	0.9942166	0.9987044
CN(0,2)	0.9938891	0.9985588
CN(0,3)	0.9839320	1.0000470
CN(0,5)	0.9678503	1.0076355
CN(0,9)	0.9479137	1.0238314
DE(1)	0.926072	1.005515

Table 4. Relative efficiencies (α_0=0.80, α_1=0.15, k=1.0)

Distribution \ Ratio	Initial value : α_0=0.80 α_1=0.15	
	R1/R2	R1/R3
SN(0,1)	0.9685382	0.999242
CN(0,2)	0.9563569	0.9980697
CN(0,3)	0.9428950	1.0017997
CN(0,5)	0.9244548	1.0095907
CN(0,9)	0.9063524	1.0604889
DExp(1)	0.8660044	1.022353

Table 5. Relative efficiencies (α_0=0.80,α_1=0.15, k=1.5)

Distribution \ Ratio	Initial value : α_0=0.80 α_1=0.15	
	R1/R2	R1/R3
SN(0,1)	0.970516	0.9982846
CN(0,2)	0.9547345	0.9936377
CN(0,3)	0.9445963	0.9938342
CN(0,5)	0.9363118	1.0043735
CN(0,9)	0.9025191	1.0367164
DExp(1)	0.8515731	0.9909657

Table 6. Relative efficiencies (α_0=0.80,α_1=0.15, k=2.0)

Distribution \ Ratio	Initial value : α_0=0.80 α_1=0.15	
	R1/R2	R1/R3
SN(0,1)	0.9793757	0.9999489
CN(0,2)	0.9683808	0.9949176
CN(0,3)	0.9560901	0.9989623
CN(0,5)	0.9474406	0.9930103
CN(0,9)	0.9103251	1.0078545
DE(1)	0.8317066	1.095708

4. Conclusion

In this simulation study, we know that the Huber type estimator works pretty better than competing estimators especially in heavy-tailed cases because there are many numbers which are less than 1in the tables. That means the estimator performs quitely well under non-normal situations. Further study should be extended to get optimal values of k which had been studied by Chan(1994) and to the more general models such as GARCH models.

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