A Wong-Zakai Type Approximation for the Multiple Itô-Wiener Integral

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ABSTRACT

We present an extension of the Wong-Zakai type approximation theorem for a multiple stochastic integral. Using a piecewise linear approximation $W^{(n)}$ of a Wiener process W, we prove that the multiple integral processes

$$\left\{\int_0^t \cdots \int_0^t f(t_1, ..., t_m) W^{(n)}(t_1) \cdots W^{(n)}(t_m), t \in [0, T]\right\},$$

where f is a given symmetric function in the space $C([0,T]^m)$, converge to the multiple Stratonovich integral of f in the uniform L^2-sense.

Keywords: Multiple Itô-Wiener integral; Multiple Stratonovich integral; Hu-Meyer's formula; Wong-Zakai theorem

1. Introduction

Our work is concerned with an extension of the basic result of Wong and Zakai (1965) to the multiple Itô-Wiener integral. The relation between ordinary integrals (or differential equations) and stochastic integrals (or stochastic differential equations) was first studied by Wong and Zakai (1965). After the work of Wong and Zakai, there have been many extensions of the Wong-Zakai theorem to multi-dimensional case and two-parameter process: Ikeda and Watanabe (1989), Twardowska (1992) and (1995), Lee, Kim and Jeon (2001).

Recently, Bardina and Jolis (2000) have studied the problem of weak convergence for the multiple Itô-Wiener integral. They considered a suitable family of approximations with absolutely continuous sample paths that converge weakly to Wiener process. When, in the multiple Itô-Wiener integral, Wiener processes are replaced by these kinds of approximations, then they proved that this ordinary

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multiple integral converges weakly to the multiple Stratonovich integral. Such familiar approximations as piecewise linear approximations or regularizations by convolution (mollifiers), among others which are considered by Bardina and Jolis (2000), converge in uniform L^2 to a Wiener process. Hence if we use these kinds of approximations we may establish a stronger approximation theorem for the multiple Itô-Wiener integral than weak convergence. In particular, the direct relationship between the multiple Itô-Wiener integral and the multiple Stratonovich integral of deterministic kernels was studied by Solé and Utzet (1990), known as Hu-Meyer's formula.

Let $(\mathbf{W}, \mathbf{P})$ be the one-parameter Wiener space equipped with the usual reference family $\{\mathcal{F}_t\}_{t\geq 0}$. The mathematical expectation in this space will be denoted by $\mathbf{E}$. For $W \in \mathbf{W}$ and each $n = 1, 2, ...,$ we set

$$W_t^{(n)} = \sum_{k=0}^{\infty} \left[W\left(\frac{i}{n}\right) + n \left[W\left(\frac{i+1}{n}\right) - W\left(\frac{i}{n}\right) \right] \left(t - \frac{i}{n}\right) \right] I_{(\frac{i}{n}, \frac{i+1}{n}]}(t).$$

As mentioned, this approximation $W^{(n)}$ converges to W in the uniform L^2-sense for any finite interval $[0, T]$, that is, $\mathbf{E}\left[\sup_{t\in[0,T]} \left|W^{(n)}(t) - W(t)\right|^2\right] \to 0$ as $n \to \infty$. Consider the following ordinary multiple integral processes

$$\left\{ J_m^{(n)}(f)_t = \int_0^t \cdots \int_0^t f(t_1, ..., t_m) W^{(n)}(t_1) \cdots W^{(n)}(t_m), t \in [0, T] \right\},$$

where f is a symmetric function in the space $\mathcal{C}([0, T]^m)$. Since $W^{(n)}$ converges, uniformly L^2, to W, we may, intuitively, expect that $J_m^{(n)}(f)$ will converge to the multiple Stratonovich integral of f in the uniform L^2-sense for any finite interval $[0, T]$.

2. Preliminaries

2.1. Multiple Itô-Wiener Integral (Itô,1951)

Let $f \in L^2([0, 1]^m)$. An elementary function $f(t_1, ..., t_m)$ is called *special* if $f(t_1, ..., t_m)$ vanishes except for the case that $t_1, ..., t_m$ are all different. We shall denote by $\mathcal{S}_m$ the totality of special elementary functions.

Theorem 2.1. (Itô) *$\mathcal{S}_m$ is a linear manifold dense in $L^2([0, 1]^m)$.*

Now we shall define the multiple Itô-Wiener integral of $f \in L^2([0,1]^m)$, which we denote by $I_m(f)$. Let f be a special elementary function. Then f can be expressible as follows:

$$f(t_1, ..., t_m) = a_{i_1,...,i_m} \quad \text{for } (t_1, ..., t_m) \in I_{i_1} \times \cdots \times I_{i_m}$$
$$= 0\text{elsewhere},$$

where $I_1, I_2, ..., I_n$ are disjoint and $a_{i_1,...,i_m} = 0$ if any two of $i_1, i_2, ..., i_m$ are equal. We define the multiple Itô-Wiener integral $I_m(f)$ for such f by

$$I_m(f) = \sum a_{i_1,...,i_m} W(I_{i_1}) \cdots W(I_{i_m}).$$

Then we may observe that the following properties hold:

(I.1) $I_m(af + bg) = aI_m(f) + bI_m(g)$, i.e., I_m is linear,

(I.2) $I_m(f) = I_m(\tilde{f})$, where $\tilde{f}(t_1, .., t_m) = \frac{1}{m!} \sum_\sigma f(t_{\sigma(1)}, ..., t_{\sigma(m)})$, σ running over all permutations of $(1, 2, ..., m)$,

(I.3) $\mathbf{E}(I_m(f) I_m(g)) = m!(\tilde{f}, \tilde{g})_{L^2([0,1]^m)}$,

(I.4) $\mathbf{E}(I_{m_1}(f) I_{m_2}(g)) = 0$, if $m_1 \neq m_2$.

Since $\mathcal{S}_m$ is dense in $L^2([0,1]^m)$ and

$$\mathbf{E}(I_m(f))^2 = m! \ \|\tilde{f}\|^2 \leq m! \ \|f\|^2 \tag{2.1}$$

by (I.3) and Cauchy-Schwarz inequality, the operator I_m can be extended uniquely to an operator on $L^2([0,1]^m)$ (denote again by I_m) which also satisfies the above properties (I.1), (I.2), (I.3), (I.4), and (**??**). For the later use, we define as $I_0(c) = c$ for any constant c.

2.2. Multiple Stratonovich integral(Solé-Utzet,1990)

Let $\Pi = \{0 = t_0^\Pi < \cdots < t_{r_\Pi}^\Pi = 1\}$ be a partition of $[0,1]$. First we introduce the following notations:

$$\Delta_i^\Pi = (t_i^\Pi, t_{i+1}^\Pi], \quad |\Delta_i^\Pi| = t_{i+1}^\Pi - t_i^\Pi,$$

$$|\Pi| = \max\{|\Delta_i^{\Pi}|, i = 0, ..., r_{\Pi} - 1\}.$$

When there is no confusion, the superindex Π in t_i and Δ_i will be omitted.

Definition. Let $f \in L^2([0,1]^m)$ be a symmetric function. We will say that f is Stratonovich integrable if there exists the limit in $L^2(\Omega)$ of

$$\sum_{i_1,..,i_m} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_m}|} \left(\int_{\Delta_{i_1} \times \cdots \times \Delta_{i_m}} f(t_1, ..., t_m) dt_1 \cdots dt_m \right) W(\Delta_{i_1}) \cdots W(\Delta_{i_m})$$

when the norm $|\Pi|$ tends to zero. This limit will be denoted by $I_m \circ (f)_1$. We will denote by $I_m \circ (f)_t$ the multiple Stratonovich integral of $f \cdot I_{[0,t]^m}$, for $t \in [0,1]$.

Definition. Let $f \in L^2([0,1]^m)$, and $j = 1, ..., [m/2]$. We will say that f has a trace of order j if the limit in $L^2([0,1]^{m-2j})$

$$\lim_{|\Pi| \downarrow 0} \sum_{i_1,...,i_j} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_j}|} \int_{\Delta_{i_1}^2 \times \cdots \times \Delta_{i_j}^2} f(t_1, ..., t_{2j}, \cdot) dt_1 \cdots dt_{2j}$$

exists. This limit will be denoted by $T_1^j f$. We will denote by $T_t^j f$, $T_1^j (f \cdot I_{[0,t]^m})$, for $t \in [0,1]$.

The following result was proved by Solé and Utzet (1990), known as Hu-Meyer's formula.

Theorem 2.5. (Solé-Utzet) *Let $f \in L^2([0,1]^m)$ be a symmetric function. If f has a trace of order j for all $j = 1, ..., [m/2]$, then f is Stratonovich integrable and*

$$I_m \circ (f)_t = \sum_{j=0}^{[m/2]} \frac{m!}{(m-2j)! j! 2^j} I_{m-2j}(T_t^j f)_t,$$

where $T_t^0 f = f \cdot I_{[0,t]^m}$.

Remark 2.6. If $f \in C([0,1]^m)$ is a symmetric function, by the continuity of f in $[0,t]^m$ it is easily seen that for all $j = 1, .., [m/2]$ the trace of order j of $f \cdot I_{[0,t]^m}$ exists and

$$T_t^j f(\cdot) = \int_{[0,t]^j} f(t_1, t_1, t_2, t_2, ..., t_j, t_j, \cdot) I_{[0,t]^{m-2j}}(\cdot) dt_1 \cdots dt_j.$$

3. Main Result

Our main result is the following theorem.

Theorem 1.1. *For every $T > 0$ and $f \in C([0,T]^m)$ a given symmetric function, we have*

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{0\le t\le T}\left|J_m^{(n)}(f)_t - I_m \circ (f)_t\right|^2\right] = 0,$$

where $I_m \circ (f)_t$ is the the multiple Stratonovich integral of f.

References

Bardina, X., Jolis, M. (2000). "Weak convergence to the multiple Stratonovich integral," *Stochastic Processes and their applications* **90**, 277-300

Ikeda, N., Watanabe, S. (1981). Stochastic Differential Equations and Diffusion Processes, Vol. 24. North-Holland Mathematical Library. North-Holland, Amsterdam.

Itô, K. (1951). "Multiple Wiener integral," *Journal of the Mathematical Society of Japan* **3(1)**, 157-169

Lee, K. S., Kim, Y. T., Jeon, J. W. (2002). "A Wong-Zakai type approximation for two-parameter processes," *Stochastic Analysis and Applications, to appear.*

Solé, J.L., Utzet, F. (1990). "Stratonovich integral and trace," *Stochastics and Stochastics Report.* **29(2)**, 203-220

Twardowska, T. (1992). "An extension of the Wong-Zakai theorem for stochastic evolution equations in Hilbert spaces," *Stochastic Analysis and Applications.* **10(4)**, 451-500.

Twardowska, T. (1995). "An approximation theorem of Wong-Zakai type for nonlinear stochastic partial differential equations," *Stochastic Analysis and Applications.* **13(5)**, 601-626.

Wong, E, Zakai, M. (1965). "On the convergence of ordinary integrals to stochastic integrals," *Annals of Statistics* **36**, 1560-1564.

Wong, E, Zakai, M. (1965). "On the relation between ordinary and stochastic differential equations," *International Joural of Engineering Society* **3**, 213-229.