On the Strong Law of Large Numbers for Arbitrary Random Variables

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ABSTRACT

For arbitrary random variables $\{X_n, n \geq 1\}$, the order of growth of the series $S_n = \sum_{j=1}^{n} X_j$ is studied in this paper. More specifically, when the series S_n diverges almost surely, the strong law of large numbers $S_n/g_n^{-1}(A_n\psi(A_n)) \to 0$ a.s. is constructed by extending the results of Petrov (1973). On the other hand, if the series S_n converges almost surely to a random variable S, then the tail series $T_n = S - S_{n-1} = \sum_{j=n}^{\infty} X_j$ is a well-defined sequence of random variables and converges to 0 almost surely. For the almost surely convergent series S_n, a tail series strong law of large numbers $T_n/g_n^{-1}(B_n\psi^*(B_n^{-1})) \to 0$ a.s., which generalizes the result of Klesov (1984), is also established by investigating the duality between the limiting behavior of partial sums and that of tail series. In particular, an example is provided showing that the current work can prevail despite the fact that the previous tail series strong law of large numbers does not work.

Keywords: arbitrary random variables, almost sure convergence of series, rate of convergence, tail series, strong law of large numbers

1. Introduction

Let $\{X_n, n \geq 1\}$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and, as usual, their partial sums are denoted by $S_n = \sum_{j=1}^{n} X_j,\ n \geq 1$. Random variables $\{X_n, n \geq 1\}$ (such that the series S_n diverges almost surely(a.s.)) are said to obey the *strong law of large numbers* (SLLN) with norming constants $\{a_n, n \geq 1\}$ if for a given sequence of positive constants with $a_n \uparrow \infty$,

$$\frac{S_n}{a_n} \to 0 \text{ a.s.}$$

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If the series S_n converges a.s. to a random variable S, then (set $S_0 = X_0 = 0$) the *tail series*

$$T_n = S - S_{n-1} = \sum_{j=n}^{\infty} X_j, \ n \geq 1$$

is a well-defined sequence of random variables and converges to 0 a.s. In the same way, random variables $\{X_n, n \geq 1\}$ are said to obey the *tail series SLLN* with norming constants $\{b_n, n \geq 1\}$ if the tail series T_n is well defined and for a given sequence of positive constants with $b_n \downarrow 0$,

$$\frac{T_n}{b_n} \to 0 \text{ a.s.}$$

In this paper, we shall be concerned with the order of growth of the series S_n whether the series converges or not, espicially, for the convergent series S_n, the rate in which S_n converges to a random variable S is equivalent to the rate in which the tail series T_n converges to 0.

For arbitrary random variables, Petrov (1973) proved SLLNs for partial sums, which are analogous to Petrov's (1969) SLLNs for partial sums of independent random variables, and Klesov (1984) developed tail series SLLNs which provided tail series analogues of Petrov's (1973) SLLNs. In this paper, it will be shown that for arbitrary random variables having no conditions on their joint distributions, Petrov's (1973) SLLNs and Klesov's (1984) tail series SLLNs are extened to wider classes of random variables by employing a class of functions instead of a specific function

$$g_n(x) = |x|^p, \ 0 < p \leq 1, \ n \geq 1. \tag{1.1}$$

2. Key Lemma

For arbitrary random variables $\{X_n, n \geq 1\}$, without the assumption of independence, the result of Petrov (1973, remark to Lemma 2) was extended in the following lemma by employing a class of functions instead of a specific function (1.1).

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be arbitrary random variables and let $\{g_n(x), n \geq 1\}$ be nondecreasing functions defined on $[0, \infty)$ such that*

$$g_n(0) = 0, \ \lim_{x \to \infty} g_n(x) = \infty \text{ and } \frac{x}{g_n(x)} \uparrow \text{ on } (0, \infty), \ n \geq 1. \tag{2.1}$$

Furthermore, let $\{b_n, n \geq 1\}$ *be a sequence of positive constants such that*

$$P\{|X_n| < b_n \text{ eventually}\} = 1. \tag{2.2}$$

If the series

$$\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(b_n)} < \infty, \tag{2.3}$$

then the series

$$\sum_{n=1}^{\infty} \frac{X_n}{b_n} \text{ converges a.s.} \tag{2.4}$$

Under the assumption that $\{X_n, n \geq 1\}$ are *independent* random variables, Lemma 2.1 has been proved by Petrov (1969, Lemma 2) who generalized the result of Chung (1947, Theorem 2).

3. Main Results

For arbitrary random variables $\{X_n, n \geq 1\}$, without assuming independence, a SLLN and a tail series SLLN are developed by applying Lemma 2.1 to partial sums and to tail series, respectively. First, in Theorem 3.1, a SLLN for arbitrary random variables is obtained by employing a class of functions instead of a specific function (1.1). As in Petrov (1969, 1973), let Ψ be the class of functions $\psi(x)$ such that each $\psi(x)$ is positive and nondecreasing for $x > x_0$ for some x_0 and the series $\sum_{n=1}^{\infty} 1/n\psi(n)$ is convergent.

Theorem 3.1. *Let* $\{X_n, n \geq 1\}$ *be arbitrary random variables and let* $\{g_n(x), n \geq 1\}$ *be strictly increasing functions defined on* $[0, \infty)$ *such that*

$$g_n(0) = 0, \lim_{x \to \infty} g_n(x) = \infty \text{ and } \frac{x}{g_n(x)} \uparrow \text{ on } (0, \infty), \ n \geq 1.$$

If $E(g_n(|X_n|)) < \infty$, $n \geq 1$, *and* $A_n = \sum_{j=1}^{n} E(g_j(|X_j|)) \uparrow \infty$, *then by assuming for some function* $\psi(x) \in \Psi$,

$$P\left\{|X_n| \leq g_n^{-1}(A_n\psi(A_n)) \text{ eventually } \right\} = 1, \tag{3.1}$$

the SLLN

$$\frac{S_n}{g_n^{-1}(A_n\psi(A_n))} \to 0 \text{ a.s.} \tag{3.2}$$

obtains, where g_n^{-1} *denotes the inverse function of* g_n *for each* $n \geq 1$.

Not only does Theorem 3.1 reduce to the result of Petrov (1973, Theorem 1) by taking (1.1), but Theorem 3.1 also yields the result of Petrov (1969, Theorem 5) by setting $g_n \equiv g,\ n \geq 1$, without assuming indenpendence in the case when the condition (A) of the theorem is assumed. Next, in Theorem 3.2, by investigating a tail series analogous result of Theorem 3.1, a tail series SLLN for arbitrary random variables is obtained. Some of Klesov's (1983, 1984) work will now be described. Let Ψ^* be the class of positive and nondecreasing functions $\psi^*(x)$ in Ψ such that $x\psi^*(x^{-1})$ tends monotonically to 0 as $x \downarrow 0$. It will be shown in the proof of Theorem 3.2 that the hypotheses ensure that $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables.

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be arbitrary random variables and let $\{g_n(x), n \geq 1\}$ be strictly increasing functions defined on $[0, \infty)$ such that*

$$g_n(0) = 0,\ \lim_{x\to\infty} g_n(x) = \infty,\ \frac{x}{g_n(x)} \uparrow \text{ on } (0,\infty),\ n \geq 1$$

and

$$g_n(x) \uparrow \text{ as } n \uparrow \text{ for each } x > 0. \tag{3.3}$$

If the series

$$\sum_{n=1}^{\infty} E(g_n(|X_n|)) < \infty, \tag{3.4}$$

then by setting $B_n = \sum_{j=n}^{\infty} E\left(g_j(|X_j|)\right),\ n \geq 1$, and assuming that for some function $\psi^*(x) \in \Psi^*$

$$P\left\{|X_n| \leq g_n^{-1}\left(B_n\psi^*(B_n^{-1})\right) \text{ eventually}\right\} = 1, \tag{3.5}$$

the tail series SLLN

$$\frac{T_n}{g_n^{-1}(B_n\psi^*(B_n^{-1}))} \to 0 \text{ a.s.} \tag{3.6}$$

obtains, where g_n^{-1} denotes the inverse function of g_n for each $n \geq 1$.

Assuming the necessary condition (3.5), then Theorem 3.2 reduces to the result of Klesov (1984, Theorem 2) by taking (1.1), and Theorem 3.2 also yields the result of Klesov (1984, Theorem 4) by setting $g_n \equiv g,\ n \geq 1$, without the independence assumption under the condition (A) of the theorem.

4. An Example

Example. Let $\{X_n, n \geq 1\}$ be arbitrary random variables (not necessarily independent) such that

$$P\left\{X_n = \frac{1}{n^2}\right\} = 1 - \frac{1}{n^2} \text{ and } P\{X_n = e^n\} = \frac{1}{n^2},\ n \geq 1.$$

Then the hypotheses of the tail series SLLN of Klesov (1984) are not met, since for any $p \in (0, 1]$

$$\sum_{n=1}^{\infty} E(|X_n|^p) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{2p}} \left(1 - \frac{1}{n^2}\right) + \frac{e^{np}}{n^2} \right\} = \infty.$$

Let $1/2 < \alpha < 1$ and let for $n \geq 1$,

$$g_n(x) \equiv (\log_1 x)^{\alpha} \text{ where } \log_1 x = \begin{cases} \log x, & \text{if } x \geq e \\ e^{-1}x, & \text{if } x < e \end{cases}.$$

Then for each $n \geq 1$,

$$E\left(g_n(|X_n|)\right) = \frac{1}{e^{\alpha}} \frac{1}{n^{2\alpha}} \left(1 - \frac{1}{n^2}\right) + \frac{n^{\alpha}}{n^2}$$

and so (3.4) holds. Now for $n \geq 1$,

$$B_n = \sum_{j=n}^{\infty} E\left(g_n(|X_j|)\right) \sim M_1\, n^{1-2\alpha} + M_2\, n^{\alpha-1}$$

$$\text{where } M_1 = \frac{1}{e^{\alpha}} \frac{1}{2\alpha - 1} \text{ and } M_2 = \frac{1}{1-\alpha}.$$

Suppose that $1/2 < \alpha \leq 2/3$. Then $B_n \sim M_1 n^{1-2\alpha}$. If $\psi^*(x)$ is taken to be the function $\psi^*(x) = (\log_1 x)^{1+\varepsilon}$ where $\varepsilon > 0$, then,

$$B_n \psi^*(B_n^{-1}) \sim M_3\, n^{1-2\alpha} (\log_1 n)^{1+\varepsilon} = o(1) \text{ where } M_3 = \frac{1}{e^{\alpha}}(2\alpha - 1)^{\varepsilon}.$$

Hence for all large n, $B_n \psi^*(B_n^{-1}) \leq 1$ implying

$$g_n^{-1}\left(B_n \psi^*(B_n^{-1})\right) = e\left(B_n \psi^*(B_n^{-1})\right)^{1/\alpha} \sim M_4\, n^{1/\alpha - 2} (\log_1 n)^{(1+\varepsilon)/\alpha}$$

where $M_4 = e{M_3}^{1/\alpha}$. Thus for all large n

$$P\left\{|X_n| > g_n^{-1}\left(B_n \psi^*(B_n^{-1})\right)\right\} = \frac{1}{n^2}$$

and so (3.5) also holds. Hence for $\varepsilon > 0$ and $\alpha \in (1/2, 2/3]$, the tail series SLLN

$$\frac{n^{2-1/\alpha}}{(\log_1 n)^{(1+\varepsilon)/\alpha}} T_n \to 0 \text{ a.s.}$$

obtains by Theorem 3.2. On the other hand, suppose that $2/3 < \alpha < 1$ then, since $B_n \sim M_2 n^{\alpha-1}$, the tail series SLLN for $\varepsilon > 0$,

$$\frac{n^{1/\alpha-1}}{(\log_1 n)^{(1+\varepsilon)/\alpha}} T_n \to 0 \text{ a.s.}$$

obtains by the same argument as was employed above.

References

(1) Chung, K.L. (1947) Note on some strong laws of large numbers, *Amer. J. Math.* **69**, 189–192.

(2) Klesov, O.I. (1983). Rate of convergence of series of random variables. *Ukrain. Mat. Zh.* **35**, 309–314 (In Russian; English translation in *Ukrainian Math. J.* **35** (1983), 264–268).

(3) Klesov, O.I. (1984). Rate of convergence of some random series. *Teor. Veroyatnost. i Mat. Statist.* **30**, 81–92 (In Russian; English translation in *Theor. Probab. Math. Statist.***30** (1985), 91–101).

(4) Petrov, V.V. (1969). On the strong law of large numbers. *Theo. Veroiatnostei i Primenen.* **14**, 193–202 (In Russian; English translation in *Theor. Probab. Appl.* **14** (1969), 183–192).

(5) Petrov, V.V. (1973). On the order of growth and sums of dependent variables, *Theo. Veroiatnostei i Primenen.* **18**, 358–361 (In Russian; English translation in *Theor. Probab. Appl.* **18** (1974), 348–350).

(6) Petrov, V.V. (1975). *Sums of Independent Random Variables*, Springer-Verlag, Berlin.