MEAN DISTANCE OF BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD[1]

by

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Abstract

Consider the mean distance of Brownian motion on Riemannian manifolds. We obtain the first three terms of the asymptotic expansion of the mean distance by means of Stochastic Differential Equation(SDE) for Brownian motion on Riemannian manifold. This method proves to be much simpler for further expansion than the methods developed by Liao and Zheng(1995). Our expansion gives the same characterizations as the mean exit time from a small geodesic ball with regard to Euclidean space and the rank 1 symmetric spaces.

Keywords: Brownian motion, Riemannian manifold, normal coordinates, Ricci curvature, Scalar curvature.

1 Introduction

Let (M,g) be an n-dimensional Riemannian manifold and (X_t, P_m) be a Brownian motion on M starting at $m \in M$. Let $\gamma_t = d(X_t, m)$ be the radial part of a Brownian motion on M where d is the distance induced by the Riemannian metric. Let T_ϵ be the first exit time from a geodesic ball of radious ϵ for the Brownian motion on M.

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Then Liao and Zheng(1995) show that

$$E(\gamma^2_{t\wedge T_\epsilon}) = nt - (1/6)\tau(m)t^2 + o(t^2), \quad \text{as} \quad t \downarrow 0,$$

where $\tau(m)$ is the scalar curvature at $m \in M$ and $o(t^2)$ may depend on ϵ. Also they prove that under some global bounded conditions,

$$E(\gamma^2_t) = nt - (1/6)\tau(m)t^2 + o(t^2), \quad \text{as} \quad t \downarrow 0$$

which does not depend on ϵ. Let Δ is the Laplace-Beltrami operator on M, that is ,

$$\Delta f = g^{ij}\nabla_i\nabla_j f,$$

where ∇_i denotes the covariant derivative $\nabla_{\partial/\partial x_i}$ with respect to the Levi-Civita connection. Liao and Zheng(1995) obtain the expansion of the mean distance using the following interesting geometric relationship;

$$\Delta^2\gamma^2(m) = -(4/3)\tau(m).$$

For the calculation of the coefficients of higher order of this expansion, we need to compute $\Delta^p\gamma^2(m)$, for $p \geq 3$. But even in case of $p = 3$, the computation will become laborious work if we follow the approach of Liao and Zheng(1995).

2 The Main Results

In this paper we develop a method to compute the coefficients of order t^3 in the power series expansion of $E(\gamma^2_t)$. Our method for asymptotic expansion of $E[\gamma^2(X_t, m)]$ turns out much simpler than the methods developed by Liao and Zheng(1995).

Before stating the our main result, we introduce the serval curvatures. We fix a normal coordinate system $(x^1, \ldots, x^n)$ about the point m. Let g_{ij}, g^{ij} and Γ^i_{jk} be the Riemannian metric, the inverse and Christoffel symbol, respectively. R_{ijkl} is the components of the curvature tensor and ρ_{ij} is the components of the Ricci curvature, that is, $\rho_{ij} = \sum_{k=1}^{n} R_{kikj}$. Also $\tau = \sum_{i=1}^{n} \rho_{ii}$ is the scalar curvature.

By definition a scalar valued curvature invariant is a polynomial in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of basis of the tangent space T_mM. Such a scalar valued invariant is said to have order k if it involves k-th derivatives of the metric tensor. Let $I(k,n)$ be a space of invariants of order $2k$ for Riemannian manifolds of dimension n. We have $\dim I(1,n) = 1$ for $n \geq 2$ and $\dim I(2,n) = 4$ for $n \geq 4$. Let $\|R\| = (\sum (R^i_{jkl})^2)^{1/2}$ and $\|\rho\| = (\sum \rho_{ij}^2)^{1/2}$ be the lengths of the curvature tensor and the Ricci curvature respectively. Then $\{\tau\}$ is a basis for $I(1,n)$ and $\{\tau^2, \|\rho\|^2, \|R\|^2, \Delta\tau\}$ is a basis for $I(2,n)$. For further information see Gray and Vanhecke(1979). Under the following global boundedness conditions given in Liao and Zheng(1995),

(1) M is compact.

(2) M is a complete Riemannian manifold with nonnegative Ricci curvature.

(3) The Ricci curvature of M is bounded from below and the exponential map $\exp_m$ at m is a diffeomorphism from T_mM onto M.

Then our main result is

$$\begin{aligned} & E[\gamma^2(X_t, m)] \\ = \; & nt - \frac{1}{6}\tau(m)t^2 \\ & + \frac{1}{90}\Big(- 6\Delta\tau(m) - \|R(m)\|^2 + \|\rho(m)\|^2 \Big) t^3 + o(t^3) \quad \text{as} \quad t \downarrow 0. \qquad (2.1) \end{aligned}$$

This expansion gives the same characterizations on Euclidean space and the rank 1 symmetric spaces as the mean exit time from a small geodesic ball used in Gray and Pinsky(1983).

References

[1] Gray, A. (1973), The volume of a small geodesic ball of a Riemannian manifold, Michigan Mat. J., 20, 329-344.

[2] Gray, A. and Pinsky, M. A. (1983), The mean exit exit time from a small geodesic ball in a Riemannain manifold, Bull.Sc.math., 107, 345-370.

[3] Gray, A. and Vanhecke, L. (1979), Riemannian geometry as determined by the volumes of small geodesic balls, Acta. math., 142, 157-198.

[4] Ikeda, N. and Watanabe, S. (1981), *Stochastic Differential Equation and Diffusion Processes*, Amsterdam/Kodansha, North-Holland, Tokyo.

[5] Liao, M. and Zheng, W. A. (1995), Radial part of Brownian motion on a Riemannian manifold, Ann. of Prob., 23(1), 173-177.

[6] Sakai, T. (1971), On eigen-values of Laplacian and curvature of Riemannian manifold, Tôhoku Math Journ., 23, 589-603.