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산란음장 해석을 위한 적분방정식에 대한 연구

A study of integral equations for the analysis of scattered acoustic field

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ABSTRACT

This paper deals with a fundamental and classical scattering problem by a finite strip. For the analysis of scattered acoustic field, a "single" integral equation is derived. Firstly, the complexity by considering the effect of the mean flow is alleviated by the introduction of Prandtl-Glauert coordinate and the new dependent variable. Secondly, the difficulty of solving the resultant strongly-coupled integral equations which always appear in this kind of 3-part mixed boundary value problem is solved by observing some good properties of the functions in complex domain and manipulating the equations and variables for the use of those properties. The solution can be obtained asymptotically in terms of gamma function and Whittaker function. One aim of this study is the improvement of methodology for the research using integral equations. The other is the basic understanding of scattering by a finite strip related to the linear cascade model of rotating fan blades.

1. Introduction

The flow and sound generated by rotating blades is one of the most important as well as difficult problems in aeroacoustics. The importance is due to the diverse application of rotating bodies in many industrial areas such as cooling fans, helicopter rotors, blowers, compressors and so on. And the difficulty is mainly because the operation is rotating motion at almost fixed position. Rotating blades can be simplified by a linear cascade model assuming that the blade is not cambered and the hub-tip ratio of the fan is close enough to unity so that curvature effects can be neglected. Thus, the rotating blades through a stationary convected disturbance can be equivalently described by the linear cascade encountering a gust with mean flow parallel to each blade [1]. The sound is calculated by solving the linearized acoustic-vorticity equations.

One remarkable feature of the mathematical description by means of partial differential equations (PDE) is the comparative ease with which solutions can be obtained for certain geometrical shapes by the method of separation of variables. In contrast, considerable difficulty is usually encountered in finding solutions for shapes not covered by the method of separation of variables.

A strip, an element of a linear cascade, is a simple shape but the solution cannot be obtained by usual methods for PDE because this finite strip yields a 3-part mixed boundary value problem (MBVP). When some parts of the boundary are prescribed by function itself and the rests of the boundary are prescribed by the normal derivative of the function, the problem is called a mixed boundary value problem. There are a few available methods (Wiener-Hopf, Riemann-Hilbert, Dual integral equation) for mixed boundary value problems. But the resultant formulas for 3-part MBVP are strongly coupled integral equations at best. Therefore, the author provides a single integral equation by decoupling these simultaneous equations. The method used in this paper is Wiener-Hopf method using the property of analytic continuation in complex domain. In section 2, some mathematical preliminaries are stated shortly without the details [1-4]. In section 3, the governing equation and boundary conditions are converted to more manageable form for the further work in complex domain. In section 4, two simultaneous integral equations are established by sum decomposition of Wiener-Hopf method. These equations are so intricate that a single one is derived by mathematical manipulations. Discussion about the solution and concluding remarks are followed in section 5.

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2. Preliminaries

2.1. Splitting Theorem [1]

Every velocity vector field u can be decomposed into solenoidal(zero divergence) and irrotational(zero curl) parts so that the pressure fluctuations are determined solely by the irrotational part.

$$u = u_1 + u_2,\ \nabla\cdot u_1 = \nabla\times u_2 = 0,\ D_t u_1 = 0,$$
$$\rho_0 D_t u_2 = -\nabla p,\ (\rho_0 c_0^2)^{-1} D_t p = -\nabla\cdot u_2$$

Since the irrotational vector u_2 is the part of the velocity associated with the pressure fluctuation, it is called the acoustical particle velocity. And since the vorticity is determined solely by the solenoidal velocity u_1, this quantity is called the vortical velocity.

2.2. Generalized Fourier Transform [2]

If $f(x)$ vanishes for $x<0$ and if $|f(x)| < Ae^{\alpha x}$ as x goes to plus infinity for some constant $A>0$, then its generalized Fourier transform is defined as

$$F_+(\lambda) = \int_0^\infty f(x)e^{i\lambda x}dx = \int_0^\infty f(x)e^{-\mathrm{Im}(\lambda)x}e^{i\mathrm{Re}(\lambda)x}dx$$

which exists and is analytic for satisfies $\mathrm{Im}(\lambda) > \alpha$. If $f(x)$ vanishes for $x>0$, and if $|f(x)| < Be^{\beta x}$ as x goes to minus infinity for some constant $B>0$, then its generalized Fourier transform is also defined as

$$F_-(\lambda) = \int_{-\infty}^0 f(x)e^{i\lambda x}dx = \int_{-\infty}^0 f(x)e^{-\mathrm{Im}(\lambda)x}e^{i\mathrm{Re}(\lambda)x}dx$$

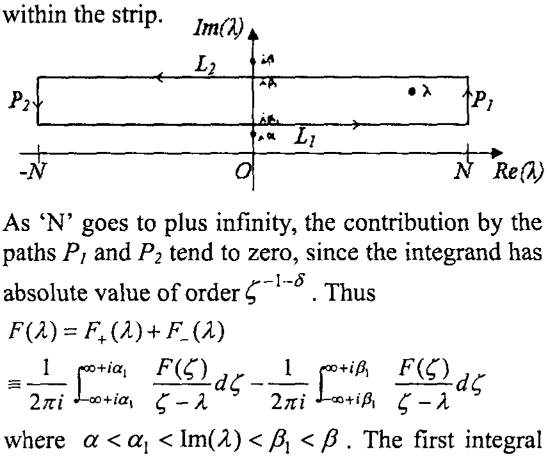
which exists and is analytic for $\mathrm{Im}(\lambda) < \beta$. Since it is always possible to choose α and β satisfying $\alpha > \beta$, there exists a region where both $F_+(\lambda)$ and $F_-(\lambda)$ are analytic. This strip bounded above by α and below by β is called an analytic strip. All the "Plus" sign means that the functions are analytic in the whole upper region bounded below by the imaginary part of the infimum of the analytic strip, and the "minus" functions are analytic in the whole lower region bounded above by the imaginary part of the supremum of the analytic strip. This analytic strip enables to find the solution with insufficient number of equations.

2.3. Wiener-Hopf method [3,4]

In some linear partial differential equations, we cannot take a Fourier transform because the boundary data type changes along the boundary. The Wiener-Hopf method is to take a Fourier transform anyway and allow part of the data to be "missing". Solving the problem using Liouville's theorem, we determine the "missing" data and the solution simultaneously. This method is applicable to linear partial differential equations on an infinite interval that have different types of boundary data on different parts of the interval.

Among the several techniques in this method, sum decomposition technique which separates a function to be analytic in each region is most important technique. Sum decomposition is not unique since we can add and subtract any polynomial (which is analytic everywhere) to both sides. Here, the basic sum decomposition technique is introduced.

Suppose $F(\lambda)$ is analytic in the strip $\alpha < \mathrm{Im}(\lambda) < \beta$ and $F = O(\lambda^{-\delta})$ as $|\lambda| \to \infty$ in the strip, for some $\delta > 0$. Then for any point λ within the strip, Cauchy's integral formula may be used to give $F(\lambda) = (2\pi i)^{-1}\int_C F(\zeta)(\zeta-\lambda)^{-1}d\zeta$ evaluated round the closed contour $C = L_1 + P_1 + L_2 + P_2$ shown in the diagram, lying within the strip.



As 'N' goes to plus infinity, the contribution by the paths P_1 and P_2 tend to zero, since the integrand has absolute value of order $\zeta^{-1-\delta}$. Thus

$$F(\lambda) = F_+(\lambda) + F_-(\lambda)$$
$$\equiv \frac{1}{2\pi i}\int_{-\infty+i\alpha_1}^{\infty+i\alpha_1}\frac{F(\zeta)}{\zeta-\lambda}d\zeta - \frac{1}{2\pi i}\int_{-\infty+i\beta_1}^{\infty+i\beta_1}\frac{F(\zeta)}{\zeta-\lambda}d\zeta$$

where $\alpha < \alpha_1 < \mathrm{Im}(\lambda) < \beta_1 < \beta$. The first integral F_+ exists and is analytic for all s such that $\mathrm{Im}(\lambda) > \alpha_1$, and the path of integration can be shifted so that α_1 is arbitrarily close to α. Similarly, the second integral F_- exists and is analytic for all $\mathrm{Im}(\lambda) < \beta_1$ arbitrarily close to β. This basic decomposition result is at the heart of the Wiener-Hopf method.

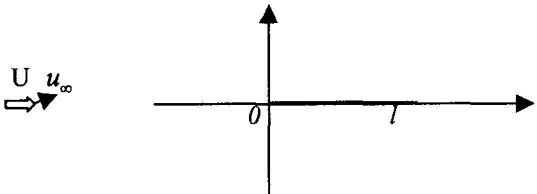
3. Problem definition

3.1. Physical Domain

A thin blade of length 'l' is encountering a small gust of u_∞ in the uniform flow U parallel to x-axis, as shown in figure. It is supposed that the vortical velocity field u_∞ is specified upstream of the blade. Due to the linearity of the problem, it is sufficient to calculate the flow field generated by a single harmonic component

$$u_\infty = \begin{pmatrix} \cos\mu \exp[i(\omega t - k(x+\cot\mu y)/M)] \\ -\sin\mu \exp[i(\omega t - k(x+\cot\mu y)/M)] \end{pmatrix}$$

where $M = U/c_0$ and $k = \omega / c_0$.



It is convenient to explicitly separate out the velocity by putting $u = u_\infty + u_s$ where u_s is the scattered velocity. Then, since u_∞ is solenoidal and satisfies $\nabla \cdot u_\infty = 0$ & $D_t u_\infty = 0$ where $D_t = \partial_t + U\partial_x$, the scattered velocity u_s satisfies $\rho_0 D_t u_s = -\nabla p$, $(\rho_0 c_0^2)^{-1} D_t p = -\nabla \cdot u_s$. These two equations are unified into a convective wave equation

$$\nabla^2 p - D_t^2 p = 0 \text{ or } \nabla^2 \varphi_s - D_t^2 \varphi_s = 0 \text{ where}$$
$$p = -\rho_0 D_t \varphi_s \text{ and } u_s = \nabla \varphi_s .$$

Since the flow is inviscid, we impose the boundary condition that the normal velocity vanishes at the surface of the blade.

$$u \cdot \vec{e}_2 = (u_\infty + u_s) \cdot \vec{e}_2 = 0$$
$$\rightarrow \ \partial_y \varphi_s \big|_{y=0} = -u_\infty \cdot \vec{e}_2 = \sin\mu \, e^{i(\omega t - kx/M)}$$

It is natural that the velocity potential also has the time factor same as incident gust, scattered potential can be expressed as $\varphi_s(x,y,t) = \varphi(x,y)e^{i\omega t}$. Then, the governing equation and the corresponding boundary condition can be rewritten as

$$(1-M)^2 \partial_x^2 \varphi + \partial_y^2 \varphi - 2ikM\varphi + k^2\varphi = 0 \qquad (1)$$
$$\partial_y \varphi \big|_{y=0} = \sin\mu \, e^{-ikx/M} \text{ for } 0 < x < l \qquad (2)$$

3.2. Prandtl-Glauert Coordinate

In order to transform this problem into an equivalent (and somewhat more familiar) stationary-medium problem, we introduce the dimensionless Prandlt-Glauert coordinates and the new dependent variable

$$\xi = x/l, \ \eta = \sqrt{1-M^2}\, y/l,$$
$$\phi(\xi,\eta) = \varphi(x,y)e^{iM\kappa\xi} \text{ where } \kappa = kl/(1-M^2).$$

Then the governing equation and the boundary conditions become

$$(\partial_\xi^2 + \partial_\eta^2 + \kappa^2)\phi = 0 \qquad (3)$$
$$\partial_\eta \phi \big|_{\eta=0} = (1-M^2)^{-1/2} l \sin\mu \, e^{i(2M-1/M)\kappa\xi}$$
$$\text{for } 0 < \xi < 1 \qquad (4)$$

κ is assumed to have a positive imaginary part κ_2, that is, $\kappa = \kappa_1 + i\kappa_2$ for the use of Wiener-Hopf method.

3.3 Complex Domain

The potential function is transformed by using generalized Fourier transform.

$$\Phi(\lambda,\eta) = \int_{-\infty}^{\infty} \phi(\xi,\eta)e^{i\lambda\xi}d\xi =$$
$$\left[\int_{-\infty}^{0} + \begin{Bmatrix} \int_0^1 \\ e^{i\lambda}\int_{-1}^{0} e^{-i\lambda} \end{Bmatrix} + e^{i\lambda}\int_1^{\infty} e^{-i\lambda} \right] \phi(\xi,\eta)e^{i\lambda\xi}d\xi$$
$$= \Phi_-(\lambda,\eta) + \begin{Bmatrix} \Phi_+^0(\lambda,\eta) \\ e^{i\lambda}\Phi_-^0(\lambda,\eta) \end{Bmatrix} + e^{i\lambda}\Phi_+(\lambda,\eta) \qquad (5)$$

Here, "plus" function is analytic for $\mathrm{Im}(\xi) > -\kappa_2$ and "minus" function is analytic for $\mathrm{Im}(\xi) < \kappa_2$. Therefore $\Phi(\xi)$ is analytic in the strip of $-\kappa_2 < \mathrm{Im}(\xi) < \kappa_2$.

Now, the Fourier transform of Eq.(3) gives a general solution

$$\Phi(\lambda,y) = \begin{cases} A_1(\lambda)e^{-\gamma(\lambda)\eta} + A_2(\lambda)e^{\gamma(\lambda)\eta}, & \eta \ge 0 \\ B_1(\lambda)e^{-\gamma(\lambda)\eta} + B_2(\lambda)e^{\gamma(\lambda)\eta}, & \eta \le 0 \end{cases}$$

where $\gamma(\lambda) = (\lambda^2 - \kappa^2)^{1/2}$.

The real part of $\gamma(\lambda)$ is always positive when the imaginary part of λ is in $-\kappa_2 < \mathrm{Im}(\lambda) < \kappa_2$. Thus, A_2 and B_1 should be zero not to diverge at infinity. And, since the normal derivatives should be continuous at $\eta = 0$, the solution is simplified to have the only one function to be determined as

$$\Phi(\lambda,\eta) = \begin{cases} A(\lambda)e^{-\gamma(\lambda)\eta}, & \eta \ge 0 \\ -A(\lambda)e^{\gamma(\lambda)\eta}, & \eta \le 0 \end{cases}. \qquad (6)$$

4. Integral Equation Formulation

Matching the Eq.(5) and Eq.(6) at $\eta = 0$ using the boundary conditions, 2-equations for 4-unknowns are constructed.

$$\Phi_-^{'}(\lambda,0) + d\frac{e^{i(\lambda-\lambda_0)}-1}{\lambda-\lambda_0} + e^{i\lambda}\Phi_+^{'}(\lambda,0) = \begin{cases} -\gamma(\lambda)e^{i\lambda}\Phi_0^-(\lambda) \\ -\gamma(\lambda)\Phi_0^+(\lambda) \end{cases} \tag{7}$$

where $d = l(1-M^2)^{-1/2}\sin\mu,\ \ \lambda_0 = (1/M - 2M)\kappa$. Since this system of equations cannot be solved by the algebra level, the complex analysis for 2-more equation is required. This is the idea of Wiener-Hopf method. $\gamma(\lambda)$ is decomposed into each analytic region by $\gamma(\lambda) = \gamma_+(\lambda)\gamma_-(\lambda) \equiv (\lambda+\kappa)^{1/2}(\lambda-\kappa)^{1/2}$.

After dividing the first equation of Eq.(7) by $e^{i\lambda}\gamma_+(\lambda)$ and the second equation of Eq.(7) by $\gamma_-(\lambda)$, the terms to be "sum decomposed" are remained. Two equations listed below are obtained by sum decomposition technique and the Liouville's theorem. The functions in the square-bracket become analytic in upper and lower half plane by the integration introduced in section 2.3.

$$\frac{\Phi_+^{'}(\lambda)}{\gamma_+(\lambda)} + \frac{de^{-i\lambda_0}}{(\lambda-\lambda_0)}\left(\frac{1}{\gamma_+(\lambda)} - \frac{1}{\gamma_+(\lambda_0)}\right) - \left[\frac{de^{-i\lambda}}{(\lambda-\lambda_0)}\frac{1}{\gamma_+(\lambda)} - \frac{e^{-i\lambda}\Phi_-^{'}(\lambda)}{\gamma_+(\lambda)}\right]_+ = 0 \tag{8.1}$$

$$\frac{\Phi_-^{'}(\lambda)}{\gamma_-(\lambda)} - \frac{d}{(\lambda-\lambda_0)\gamma_-(\lambda)} + \left[\frac{de^{i(\lambda-\lambda_0)}}{(\lambda-\lambda_0)\gamma_-(\lambda)} + \frac{e^{i\lambda}\Phi_+^{'}(\lambda)}{\gamma_-(\lambda)}\right]_- = 0 \tag{8.2}$$

where

$$[S(\lambda)]_- = -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{S(\alpha)}{\alpha-\lambda}d\alpha\ ;\ [S(\lambda)]_+ = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{S(\alpha)}{\alpha-\lambda}d\alpha$$

and $-\kappa_2 < a < \mathrm{Im}(\lambda) < b < \min(1/M - 2M, 1)\kappa_2$.

For $M<0.5$, $\min(1/M-2M,1)=1$, so we can choose 'c' such that $c = b = -a\ (c>0)$ and let $\alpha = -\alpha$ in the integrand of the Eq.(8.1) and $\lambda = -\lambda$ in the Eq.(8.2). Using the property of $\gamma_+(-\alpha) = \gamma_-(\alpha)$, interchanging the path of integration, adding and subtracting two equations gives a single integral equation. This is the key result whose detail procedure cannot be shown due to its length.

$$\frac{\Theta_{1,2}(\lambda)}{\sqrt{\lambda-\kappa}} \pm \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{e^{i\alpha}}{\alpha+\lambda}\frac{\Theta_{1,2}(\alpha)}{\sqrt{\alpha+\kappa}}d\alpha = R(\lambda) \tag{9}$$

where $\Theta_{1,2}(\lambda) = \left\{\Phi_+^{'}(\lambda) \pm \Phi_-^{'}(-\lambda)\right\} + d\left\{\frac{e^{-i\lambda_0}}{\lambda-\lambda_0} \pm \frac{1}{\lambda+\lambda_0}\right\}$,

$\mathrm{Im}(-\lambda) < c < \kappa_2$ and $R(\lambda) = \dfrac{de^{-i\lambda_0}}{\sqrt{2\lambda_0(\lambda-\lambda_0)}}$.

5. Concluding Remark

A single integral equation is derived for a scattering by a finite strip. And the solution can be obtained by using the asymptotic evaluation of integral of the form

$$I = \int_{-\infty+ic}^{\infty+ic}\frac{e^{-i\alpha}}{\alpha+\lambda}\frac{\Theta(\alpha)}{\sqrt{\alpha+\kappa}}d\alpha$$

where $\mathrm{Im}(-\lambda) < c < \kappa_2$.

We can write

$$\Theta(\alpha) = (\alpha-p)^{r+1/2}f(\alpha) = (\alpha-p)^{r+1/2}\left\{f(q) + (\alpha-q)f'(q) + \ldots\right\}$$

where $r = -1, 0, 1, 2, \ldots$ and $f(\alpha)$ can be expanded as a Taylor series about some 'q'.

Then $I \sim 2e^{i(\kappa+\pi/4)}i^r\left[f(\kappa)W_r(z)\right]$

where $z = -i(\lambda+\kappa)$

and $W_{n-1/2}(z) = \Gamma(n+1)e^{z/2}z^{(n-1)/2}W_{-(n+1)/2,n/2}(z)$

where $\Gamma(z)$ is a gamma function and $W_{i,j}(z)$ is a Whittaker function.

It is possible to improve the treatment of the equation by using more suitable types of asymptotic expansions. And also the local behavior for large or small wave numbers will be investigated.

References

(1) Goldstein, M. E., Aeroacoustics, McGraw-Hill, 1976

(2) Crighton, D. G., et al, Modern methods in analytical acoustics, Springer-Verlag, 1994

(3) Zwillinger, D., Handbook of differential equations, Academic Press, 1997

(4) Noble, B., Methods based on the Wiener-Hopf technique for the solution of partial differential equations, International series of monographs on pure and applied mathematics, Pergamon Press, 1958