Analysis of Leaky Modes on Circular Dielectric Rods using Davidenko' s Method

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Abstract

Leaky modes on a circular dielectric rod are investigated from the precisely determined normalized complex propagation constants using Davidenko' s complex root finding technique. Below the cutoff frequency of the guided mode, distinct frequency regions that have unique properties are observed, such as nonphysical region, antenna mode region, reactive mode region, and spectral gap region. The effects of two design parameters, dielectric constants and the radius of the rod, to the leaky mode characteristics are also considered.
Key words: leaky mode, circular dielectric rod, complex propagation constant, Davidenko' s method.

I. INTRODUCTION

A circular dielectric rod is one of the simplest structures that can be used as either a waveguide or an antenna. Above the guided mode cutoff, the rod operates as a waveguide and its modal characteristics have been studied extensively for a long time [1, 2], even nowadays [3]. The propagation constants of the guided mode are purely real if the dielectric material is assumed to be lossless. Below the cutoff frequency of the guided modes, in radiation mode region [4], however, the propagation constants become complex and make up another class of discrete set of eigenvalue solutions, which represents leaky modes. The imaginary term in the complex propagation constants of the leaky modes represents the leakage of the guided power to the free space region. The major feature of the leaky modes is that the amplitude of the waves is growing along the transverse direction, so it is called the improper waves [5]. The phase constant and the attenuation constant in the complex propagation constant, which represents the dispersion of the leaky modes, are two of the most important parameters that characterize the electromagnetic structures. The leaky mode dispersion characteristics of the circular dielectric rod are little known, in spite of that its characteristic equation has been revealed long time ago [6]. Since the circular dielectric rod is easy to design and fabricate and has much potential to be more widely used especially in the leaky mode region, e.g., an omni-directional antenna, the characteristics of the leaky modes are more to be analyzed below the cutoff of the guided mode. In 1969, Arnbak determined the complex propagation constants of the leaky modes of the circular dielectric rod and showed the existence of the leaky modes in a circular dielectric rod applying an approximate analysis to the characteristic equation [7]. However, it is focused on finding the complex propagation constants and more detailed discussions about the leaky modes of the dielectric rod were not available at that time.

In this paper, we investigated the leaky modes of the dielectric rod in more detail such as the nonphysical leaky mode, the antenna mode, the reactive mode, and the spectral gap for the lowest several TM modes from the precisely determined normalized complex propagation constants by the Davidenko's complex root finding technique, which is known to be robust in its insensitiveness of initial guesses. The effects of the two design parameters, the radius of the dielectric rod and the dielectric constant of the rod material, to the dispersion relation are also considered.

II. CHARACTERISTIC EQUATION

Fig. 1 shows the structure of the circular dielectric rod employed in this work. The axial component of the electric and the magnetic fields can be expressed as follows.

$$E_{zd} = A_{mn} J_m(k_d r)\exp[j(\omega t - m\theta - \gamma z)] \quad (1)$$

$$H_{zd} = B_{mn} J_m(k_d r)\exp[j(\omega t - m\theta - \gamma z)] \quad (2)$$

$$E_{zf} = C_{mn} H_m^{(2)}(k_f r)\exp[j(\omega t - m\theta - \gamma z)] \quad (3)$$

$$H_{zf} = D_{mn} H_m^{(2)}(k_f r)\exp[j(\omega t - m\theta - \gamma z)] \quad (4)$$

J_m and $H_m^{(2)}$ are the mth order Bessel and Hankel functions of the second kind; m is the azimuthal eigenvalue; A_{mn}, B_{mn}, C_{mn}, and D_{mn} are complex constants

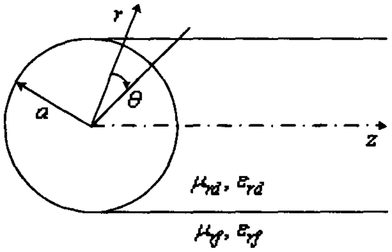

Fig. 1 Circular dielectric rod

corresponding to the modes; k_d and k_f are the complex transverse propagation constants in the dielectric region and the free space region, respectively, and are related with the material constants and the complex axial propagation constants as

$$k_i^2 = k_0^2\mu_{ri}\varepsilon_{ri} - \gamma^2 = k_0^2\left(\mu_{ri}\varepsilon_{ri} - \bar{\gamma}^2\right),\ (i = d, f) \tag{5}$$

Here, k_0 is the free space wave number, and $\bar{\gamma}$ is the normalized complex axial propagation constant, which is composed of the normalized phase and the attenuation constants, i.e.,

$$\bar{\gamma} = \frac{\gamma}{k_0} = \frac{\beta - j\alpha}{k_0} = \frac{\beta}{k_0} - j\frac{\alpha}{k_0} = \bar{\beta} - j\bar{\alpha} \tag{6}$$

In the case of leaky modes ($\bar{\alpha} \neq 0$), the following relationship should be satisfied, which is by substituting (6) into (5) [5].

$$\begin{aligned} &\left(\mathrm{Re}\{k_i\}\right)^2 - \left(\mathrm{Im}\{k_i\}\right)^2 = k_0^2\left(\mu_{ri}\varepsilon_{ri} - \bar{\beta}^2 + \bar{\alpha}^2\right) \\ &\mathrm{Re}\{k_i\}\mathrm{Im}\{k_i\} = k_0^2\bar{\alpha}\bar{\beta} \qquad (i = d, f) \end{aligned} \tag{7}$$

where $\mathrm{Re}\{k_i\}$ and $\mathrm{Im}\{k_i\}$ are the real and the imaginary parts of the transverse propagation constants, respectively, and both parts are positive. Applying the boundary conditions at the radius of the circular dielectric rod $r = a$ to the axial and the azimuthal components of the fields in the dielectric and the free space region yields the 4×4 coefficient matrix, and the determinant of the matrix should be zero to avoid nontrivial solutions. This is the characteristic equation of the circular dielectric rod and is expressed as follows.

$$P^2 - QR = 0 \tag{8}$$

$$P = m\left(\frac{\bar{\gamma}}{a}\right)\left(\frac{1}{k_d^2} - \frac{1}{k_f^2}\right) \tag{9}$$

$$Q = \frac{\varepsilon_{rd}}{k_d}\frac{J_m'(k_d a)}{J_m(k_d a)} - \frac{\varepsilon_{rf}}{k_f}\frac{H_m^{(2)\prime}(k_f a)}{H_m^{(2)}(k_f a)} \tag{10}$$

$$R = \frac{\mu_{rd}}{k_d}\frac{J_m'(k_d a)}{J_m(k_d a)} - \frac{\mu_{rf}}{k_f}\frac{H_m^{(2)\prime}(k_f a)}{H_m^{(2)}(k_f a)} \tag{11}$$

For $m = 0$, the characteristic equation (8) is decoupled to the characteristic equation of the TM_{0n} mode (Q=0), and the TE_{0n} mode (R=0), respectively. As seen in (10) and (11), the characteristic equations of the TM_{0n} and the TE_{0n} mode are identical except for the material constants and the hybrid mode such as the HE_{mn} and EH_{mn} modes are linear combinations of each transverse mode, thus, the TM_{0n} mode represents the most general feature of the circular dielectric rod in our case. So, we focus on our attention to the characteristics of the TM_{0n} modes. The characteristic equation of the TM_{0n} mode can be expressed as follows.

$$Q = \frac{\varepsilon_{rd}}{k_d}\frac{J_1(k_d a)}{J_0(k_d a)} - \frac{\varepsilon_{rf}}{k_f}\frac{H_1^{(2)}(k_f a)}{H_0^{(2)}(k_f a)} = 0 \tag{12}$$

III. DAVIDENKO'S COMPLEX ROOT FINDING TECHNIQUE

Below the cutoff frequency of the guided modes, the propagation constants become complex, and the normalized phase and attenuation constants in (7) are two of the most important parameters in analyzing leaky mode characteristics and designing leaky mode antennas. Thus, normalized complex propagation constants should be determined precisely. The complex roots (the complex propagation constants) in (12) are determined with Davidenko's complex root finding technique [8]. The Davidenko's method of complex roots finding algorithms is known to be superior to another complex roots finding methods such as the Müller's method [9] or Newton's method in its insensitiveness of initial guess and high speed of root search. In principle, the Davidenko's method transforms n-coupled nonlinear algebraic equations with n unknowns into n-coupled first order ordinary differential equations with a dummy scalar variable. As the dummy variable goes to infinity, each variable approaches to true values. Note that the Davidenko's method can be applied only in the case of analytic functions. In our case of searching the values of the normalized phase and attenuation constants, eq. (12) is transformed as follows:

$$\begin{cases} \dfrac{d\bar{\beta}}{dt} = -\dfrac{\mathrm{Re}[Q]\mathrm{Re}\left[Q_{\bar{\gamma}}\right] + \mathrm{Im}[Q]\mathrm{Im}[Q_{\bar{\gamma}}]}{\left|Q_{\bar{\gamma}}\right|^2} \\ \dfrac{d\bar{\alpha}}{dt} = \dfrac{\mathrm{Re}[Q]\mathrm{Im}\left[Q_{\bar{\gamma}}\right] - \mathrm{Im}[Q]\mathrm{Re}[Q_{\bar{\gamma}}]}{\left|Q_{\bar{\gamma}}\right|^2} \end{cases} \tag{13}$$

Q is the characteristic equation of the TM_{0n} mode in eq. (12) and it can be considered as a function of the normalized phase and attenuation constants, *i.e.*, $Q(\bar{\beta},\bar{\alpha})=0$. $Q_{\bar{\gamma}}$ is the total derivative of the Q with respect to the normalized propagation constant $\bar{\gamma}$, that is

$$Q_{\bar{\gamma}}=\frac{dQ}{d\bar{\gamma}}=\mathrm{Re}\left[\left\{\frac{\partial}{\partial\bar{\beta}}-j\frac{\partial}{\partial\bar{\alpha}}\right\}Q\right] \quad (14)$$

since Q is an analytic function. A system of two coupled first order ordinary differential equation with a dummy variable t, eq. (13), is implemented with MATHEMATICA 4.0 and numerically solved. The obtained normalized phase and attenuation constants are substituted to the original *TM* mode characteristic equation and are checked the accuracy of the returned value compared with zero, achieving the tolerances under 10^{-10} for both the real parts and the imaginary parts. The obtained normalized complex phase and attenuation constants are also substituted in (7), and have checked that the modes have these values are the (forward) leaky modes or not [5].

IV. NUMERICAL RESULTS

At first, we consider the normalized complex propagation constants of the circular dielectric rod for the three lowest order modes. The dielectric constant and the radius of the dielectric rod are arbitrarily chosen to be 5.0 and 5.0mm, respectively. Below the cutoff frequencies of the guided modes, several kinds of distinct frequency regions such as the nonphysical mode regions, radiation mode regions, antenna mode region, and spectral gap regions are observed. Fig. 2 shows the normalized phase constants of the leaky modes for TM_{01}, TM_{02}, and TM_{03} modes as well as the guided modes and Fig. 3 shows the corresponding normalized attenuation constants. The cutoff frequencies of the guided modes for the TM_{01}, TM_{02}, and TM_{03} modes are 11.48, 26.35, and 41.32 GHz, respectively and below these cutoff frequencies, the nonzero value of the normalized attenuation constants are introduced. As the operating frequency approaches to zero, the normalized phase constants exceed the unity. In this range, the normalized phase constants do not have physical meaning [10]. They approaches to infinity as the frequency goes to zero as well as the normalized attenuation constants do, as shown in Fig. 3. The upper limits of these nonphysical mode regions for the TM_{01}, TM_{02}, and TM_{03} are 3.51, 1.98, and 1.95 GHz, respectively. Above these limit frequencies, the normalized phase constants decrease to the minimum values and then increase to the unity again. This frequency region corresponds to the physical leaky mode

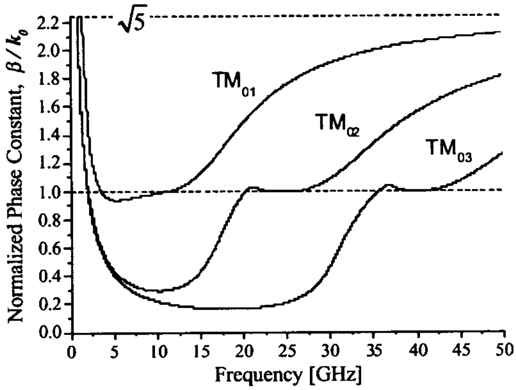

Fig. 2 Normalized phase constants.

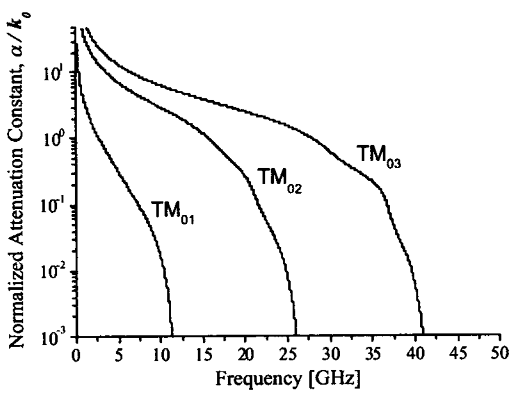

Fig. 3 Normalized attenuation constants.

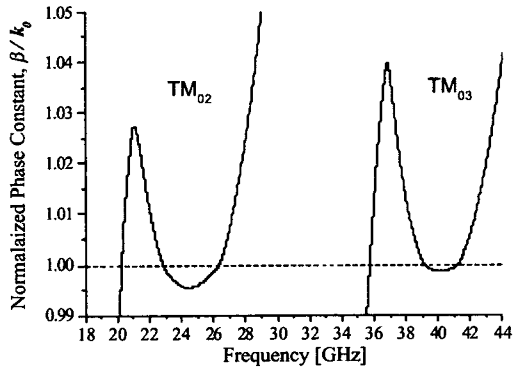

Fig. 4 Normalized phase constants with enlarged scale near the unity.

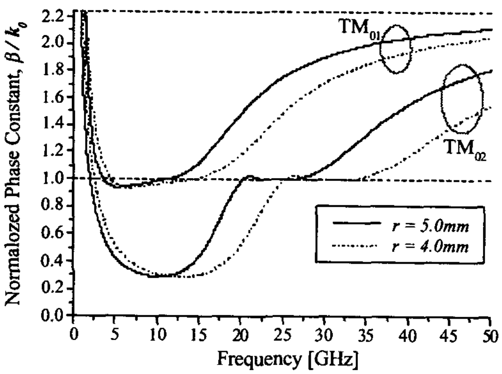

Fig. 5 Normalized phase constants when the dielectric constant of the dielectric rod is 5.0 with different radius of the circular dielectric rod.

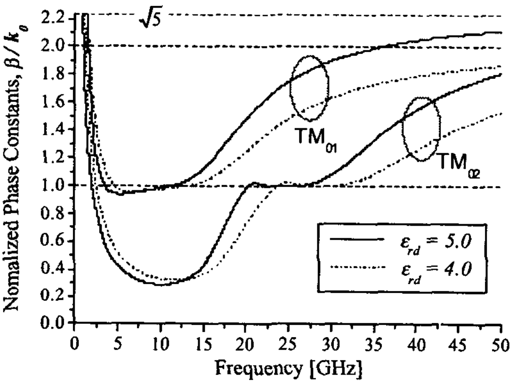

Fig. 6 Normalized phase constants when the radius of the dielectric rod is 5.0 mm with different dielectric constant.

regions [11]. This region can be divided into two distinct regions as the reactive mode regions and the antenna mode regions; if the normalized phase constant is greater than the normalized attenuation constants, the mode is referred to as the antenna mode region ($\bar{\beta} < 1$, $\bar{\beta} > \bar{\alpha}$) and if not, the reactive mode region ($\bar{\beta} > 1$, $\bar{\beta} < \bar{\alpha}$) [12]. The frequency of which the normalized phase and attenuation constants have same value is apt to be called also cutoff. It means the cutoff of the leakage of power to the free space and at this frequency the quantity of the guiding real and imaginary powers are same. This frequency always lies above the frequency at which the minimum point of the normalized phase constants. Especially for the TM_{01} mode, since the frequency that the normalized phase constants

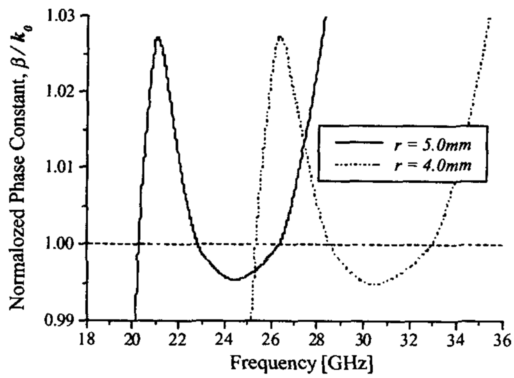

Fig. 7 Normalized phase constants with enlarged scale near the unity when the dielectric constant of the dielectric rod is 5.0 with different radius of the circular dielectric rod for the TM_{02} mode.

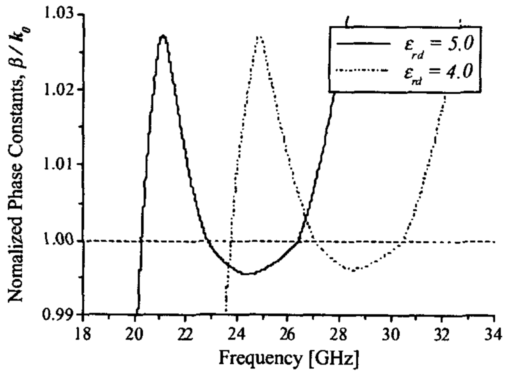

Fig. 8 Normalized phase constants with enlarged scale near the unity when the radius of the dielectric rod is 5.0 mm with different dielectric constant for the TM_{02} mode.

and the normalized attenuation constants are same lies on the nonphysical mode region that we previously mentioned, the TM_{01} mode does not have the reactive mode region. The reactive mode regions for the TM_{02} and TM_{03} modes are ranging from 1.98 to 17.15 GHz with 15.17 GHz width and from 1.95 to 30.57 GHz with 28.62 GHz width, respectively; the antenna mode regions for the TM_{02} and TM_{03} modes are from 17.15 GHz to 20.27 GHz with 3.12 GHz width and 30.57 to 35.76 GHz with 5.19 GHz width, respectively. Both the widths of the reactive mode region and the antenna mode region are increased as higher the modes. The antenna mode region for the TM_{01} mode is reserved for another antenna mode region, which is going to be mentioned. Fig. 4 shows the normalized phase constants near the unity corresponding to Fig. 2. As the frequency goes to higher than that of the antenna mode region, the normalized phase constants exceed the unity again, seen from Fig. 4. This region also has no physical meaning and is called the spectral gap region, ranging from 20.27 to 22.84 GHz with 2.57 GHz width and from 35.76 to 39.13 GHz with 3.37 GHz width for the TM_{02} and TM_{03} modes, respectively. The width of the spectral gap region increases as higher the modes. Note that the TM_{01} mode has no spectral gap regions. The remaining portion of the frequency regions below the cutoff frequency of the guided mode is another antenna mode region above the spectral gap region in frequency, ranging from 3.51 to 11.48 GHz with 7.97 width, from 22.84 to 26.36 GHz with 3.52 GHz width, and from 39.13 to 41.32 GHz with 2.19 GHz width for the TM_{01}, TM_{02}, and TM_{03} mode, respectively. The width of the second antenna mode shrinks as higher the modes. The upper limit frequency of this range meets the cutoff frequency of the guided mode. In other guiding structures such as the NRD guide [12] and the partially dielectric-loaded open guiding structure [13], the normalized attenuation constant becomes zero at the frequency with maximum normalized phase constants within the spectral gap region. The normalized attenuation constant of our structure becomes zero at this cutoff frequency of the guided mode, however, outside the spectral gap region, implying that the spectral gap region is not always consistent with the transition region between the guided mode and the leaky mode region. Next, we investigated the effects of the radius and the dielectric constants of the dielectric rod to the dispersion. Fig. 5 and Fig. 6 show the normalized phase constants of the circular dielectric rod with the same dielectric constant and different radius of the rod, respectively. The upper limit frequency of the nonphysical mode and the cutoff frequency of the guided mode are shifted to a higher frequency as the radius of the rod and the dielectric constant of the rod are decreased in the case of the TM_{01} and TM_{02} modes. Fig. 7 and Fig. 8 are the enlarged scale of the normalized phase constants near the unity, corresponding to Fig. 5 and Fig. 6, respectively. The widths of the reactive mode region, the antenna mode region, and the spectral gap region are widened for the TM_{02} mode. The width of the second antenna mode is shortened as the dielectric constant of the rod is decreased; however, it is widened as the radius of the rod is decreased.

V. CONCLUSION

We investigated the leaky modes on a circular dielectric rod structure from the precisely determined normalized phase and attenuation constants by Davidenko' s method. In the frequency region below the cutoff of the guided modes, distinct frequency regions such as nonphysical region, antenna mode region, reactive mode region, spectral gap regions are observed. The effects of the dielectric constant and the radius of the rod to the dispersion are considered.

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