Spline Surface Approximation for Computing Pit Excavation Volume with the Free Boundary Conditions†

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Abstract

The calculation of earthwork plays a major role in plan or design of many civil engineering projects, and thus it has become very important to improve the accuracy of earthwork calculation. In this paper, we propose an algorithm of finding a cubic spline surface with the free boundary conditions, which interpolates the given three dimensional data, by using B-spline and an accurate method to estimate pit-excavation volume. The proposed method should be of interest to surveyors especially concerned with accuracy of volume computations. We present some computational results showing that our proposed method provides good accuracy.

Key words. Pit-excavation volume, spline, B-spline, spline surface

1. Introduction

The estimation of pit excavation volume is of interest in many surveying and highway applications. Several methods have been developed for estimating the pit excavation volume ranging from a simple formula to more complicated formulas and numerical methods. The standard methods can be characterized with three basic ideas such as trapezoidal rule, Simpson rule and cubic spline function. The trapezoidal method, which is the most simple method, approximates the ground profile of each grid cell by a plane and estimates the pit excavation volume as the product of the area of the grid cell and the average excavation heights of the grid cell corners (Anderson et al, 1985; Schmidt and Wong 1985; Wolf and Brinker 1989; Moffit and Bossler 1998). This method is the most in common, but the

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interfaces between the approximating planes are sharp and it may not properly describe the ground surface. The Simpson based methods improve the accuracy of the volume estimation for the approximation of the ground surface by considering a second-degree polynomial or a third-degree polynomial in each direction of the grid (Easa 1988; Chambers 1989). In Easa (1988), it was assumed that the rectangles formed by the grid were of equal size-that is, the grid was formed by taking equal size intervals along each of the axes. Chambers (1989) generalized Easa's result by allowing grids in which the rectangles were of unequal sizes-that is, the grid were formed by partitioning the axes into intervals of unequal sizes. But both methods have a common drawback: the interfaces of the approximating surfaces are sharp. To eliminate this drawback, Chen and Lin (1991) proposed the cubic spline method, which provides smooth connections between the approximating cubic spline polynomials with the natural boundary conditions. Also, Easa (1998) developed the cubic Hermite polynomial method, which guarantees smooth connections between the approximating cubic Hermite polynomials. Yoo, Lee and Mun (2002) extended Chen and Lin's idea to the three dimensional problem and developed the method providing an interpolating cubic spline surface with the natural boundary conditions.

In this paper, we propose a method of finding a cubic spline surface with the free boundary conditions, which interpolates the given three dimensional data, by using cubic B-splines. Our proposed method approximates the ground surface with the cubic spline polynomial with the free boundary conditions along both x and y directions. The method is based on the cubic B-spline, and the interpolating cubic spline surface can be obtained by those B-splines. We describe basic properties of spline and B-spline without proofs in section 2. We also introduce a piecewise cubic spline surface with the free boundary conditions interpolating given data and its induced linear systems in section 3. Computational results of the proposed method and some comments are presented in section 4.

2. Basic results of spline and B-spline

In this section, we describe basic properties of spline and B-spline without proofs. For the details of proofs, refer to Farin (1988) and Lyche and Morken (1999).

Let two points $c_1=(x_1, y_1)$ and $c_2=(x_2, y_2)$ be given. Then the line segment joining above two points can be expressed as

$$p(t|c_1, c_2; t_2, t_3) = \frac{t_3-t}{t_3-t_2}c_1 + \frac{t-t_2}{t_3-t_2}c_2, \quad t\in[t_2, t_3]. \tag{1}$$

The two parameters t_2 and t_3 are arbitrary real numbers with $t_2 < t_3$. Regardless of how we choose the parameters, the resulting curve is always same. The construction of a piecewise linear curve based on some given points $(c_i)_{i=1}^{n}$ is quite obvious; we just connect each pair of points by a straight line. More specifically, we choose n numbers $(t_i)_{i=2}^{n+1}$ with $t_i < t_{i+1}$ for $i=2,3,\cdots,n$, and define the curve f by

$$f(t)=\begin{cases} p(t|c_1, c_2; t_2, t_3) & t\in[t_2, t_3), \\ p(t|c_2, c_3; t_3, t_4) & t\in[t_3, t_4), \\ \vdots & \\ p(t|c_{n-1}, c_n; t_n, t_{n+1}) & t\in[t_n, t_{n+1}) \end{cases} \tag{2}$$

The points $(c_i)_{i=1}^{n}$ are called the control points of the curve, while the parameters $t=(t_i)_{i=2}^{n+1}$, which give the value of t at the control points, are referred to as the knots, or knot vector, of the curve. If we introduce the piecewise constant functions

$$B_{i,0}(t)=\begin{cases} 1, & t_i \le t < t_{i+1}, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

and set $p_{i,1}(t)=p(t|c_{i-1}, c_i; t_i, t_{i+1})$, we can write $f(t)$ in (2) more precisely as

$$f(t)=\sum_{i=2}^{n} p_{i,1}(t)\,B_{i,0}(t) \tag{4}$$

This construction can be generalized to produce smooth, piecewise polynomial curves of higher degrees. For the quadratic spline curve, let three control points c_1, c_2, c_3 be given and set the knots with $t_2 \le t_3 < t_4 \le 5_5$. Then we can obtain the quadratic spline curve by using two straight lines passing through c_1 and c_2, c_2 and c_3 in the following way.

$$p(t|c_1, c_2, c_3; t_2, t_3, t_4, t_5)=\frac{t_4-t}{t_4-t_3}\,p(t|c_1, c_2; t_2, t_4)+\frac{t-t_3}{t_4-t_3}\,p(t|c_2, c_3; t_3, t_5). \tag{5}$$

Here t is the parameter which is in $[t_3, t_4]$. For any n control points $(c_i)_{i=1}^{n}$, we can define the piecewise quadratic spline curve by using the formula (5) and the knot vector $(t_i)_{i=2}^{n+2}$ with $t_2 \le t_3 < \cdots < t_{n+1} \le t_{n+2}$.

$$f(t) = \begin{cases} p(t \mid c_1, c_2, c_3; t_2, t_3, t_4, t_5), & t_3 \le t \le t_4, \\ p(t \mid c_2, c_3, c_4; t_3, t_4, t_5, t_6), & t_4 \le t \le t_5, \\ \vdots & \\ p(t \mid c_{n-2}, c_{n-1}, c_n; t_{n-1}, t_n, t_{n+1}, t_{n+2}), & t_n \le t \le t_{n+1}. \end{cases} \tag{6}$$

Set $p_{i,2}(t) = p(t \mid c_{i-2}, c_{i-1}, c_i; t_{i-1}, t_i, t_{i+1}, t_{i+2})$. Then we can write the formula (6) more precisely as

$$f(t) = \sum_{i=3}^{n} p_{i,2}(t) B_{i,0}(t) \tag{7}$$

Similarly, we can define the piecewise cubic spline curve.

$$f(t) = \sum_{i=4}^{n} p_{i,3}(t) B_{i,0}(t), \tag{8}$$

where, for the parameter $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} p_{i,3}(t) &= p(t \mid c_{i-3}, c_{i-2}, c_{i-1}, c_i; t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, t_{i+3}) \\ &= \frac{t_{i+1} - t}{t_{i+1} - t_i} p(t \mid c_{i-3}, c_{i-2}, c_{i-1}; t_{i-2}, t_{i-1}, t_{i+1}, t_{i+2}) \\ &\quad + \frac{t - t_i}{t_{i+1} - t_i} p(t \mid c_{i-2}, c_{i-1}, c_i; t_{i-1}, t_i, t_{i+2}, t_{i+3}). \end{aligned}$$

The formulas (4), (7), and (8) can also be written in the form of

$$f(t) = \sum_{i=1}^{n} c_i B_{i,d}(t) \tag{9}$$

where $B_{i,d}(t)$ is given by the recurrence relation

$$B_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} B_{i,d-1}(t) + \frac{t_{i+1+d} - t}{t_{i+1+d} - t_{i+1}} B_{i+1,d-1}(t), \quad d = 1,2,3. \tag{10}$$

Here the function $B_{i,d}$ is called a B-spline of degree $d(d=1,2,3)$ with knots $\boldsymbol{t}$. In (10) possible divisions by zero are resolved by the convention that 'anything divided by zero is zero'. Note that by the formula (10) we can obtain a B-spline of degree d when an appropriate knot vector is given, and get a piecewise spline of degree d with a linear combination of B-splines. The B-spline $B_{j,d}$ depends only on the knots $(t_i)_{i=1}^{j+d+1}$. To understand the nature of B-splines, $B_{j,d}(t) = B(t|t_j, \cdots, t_{j+d+1})$ is sometimes useful. For example, if $d \geq 2$ and if we set $(t_j, \cdots, t_{j+d}, t_{j+d+1}) = (a, b, \cdots, c, d)$, then (10) can be written

$$B(t\,|a, b, \cdots, c, d)(t) = \frac{t-a}{c-a} B(t|a, b, \cdots, c)(t) + \frac{d-t}{d-b} B(t|b, \cdots, c, d)(t). \tag{11}$$

For any given m points $(x_i, y_i)_{i=1}^{m}$, we have to choose an appropriate knot vector in order to find a piecewise spline $f(x)$ in an appropriate piecewise spline space such that $f(x_i) = y_i$ for $i = 1,2,\cdots,m$. We use the Schoenberg-Whitney nesting conditions, see Farin (1988) and Lyche and Morken (1999), on the knot vector for the existence and uniqueness of the piecewise spline $f(x)$. For example, if we want to find a piecewise cubic spline $f(x)$, $(x_1, x_1, x_1, x_1, x_2, \cdots, x_{m-1}, x_m, x_m, x_m, x_m)$ is an appropriate knot vector for the natural or Hermite boundary conditions and $(x_1, x_1, x_1, x_1, x_3, x_4, \cdots, x_{m-2}, x_m, x_m, x_m, x_m)$ is suitable for the free boundary conditions.

Note that there are many ways to determine an appropriate knot vector satisfying the Shoenberg-Whitney nesting conditions for a piecewise cubic spline surface interpolating the given data. In this paper, we only concentrate on the piecewise cubic spline with the free boundary conditions. We just take a knot vector with $(x_1, x_1, x_1, x_1, x_3, x_4, \cdots, x_{m-2}, x_m, x_m, x_m, x_m)$ for the free boundary conditions satisfying the Schoenberg-Whitney nesting conditions which generates the number of m B-splines. Then the m by m linear system induced by the interpolation problem can be solved uniquely by the choice of the above knot vector. We can extend this idea to the three dimensional problem in the next chapter.

3. Spline surface interpolation and its induced linear systems

We consider an interpolation problem at a set of gridded data $(x_i, y_i, f_{ij})_{i=1,j=1}^{m_1,m_2}$, where $a = x_1 < x_2 < \cdots < x_{m_1} = b$ and $c = y_1 < y_2 < \cdots < y_{m_2} = d$. For each i,j, we can think of f_{ij} as the value of an unknown function $f = f(x, y)$ at the point (x, y).

We want to find a method to find a piecewise cubic spline surface $g = g(x, y)$ satisfying the free boundary conditions in a tensor product space $S_1 \otimes S_2$ such that $g(x_i, y_j) = f_{ij}$ where $i = 1, \cdots, m_1$ and $j = 1, \cdots, m_2$.

For the free boundary conditions in the three dimensional problem, we just take the knot vectors with $(x_1, x_1, x_1, x_1, x_3, x_4, \cdots, x_{m_1-2}, x_{m_1}, x_{m_1}, x_{m_1}, x_{m_1})$ and $(y_1, y_1, y_1, y_1, y_3, y_4, \cdots, y_{m_2-2}, y_{m_2}, y_{m_2}, y_{m_2}, y_{m_2})$ for the x-direction and the y-direction respectively. We think of S_1 and S_2 as two univariate piecewise cubic spline spaces $S_1 = \text{span}\{\phi_1, \cdots, \phi_{m_1}\}$ and $S_2 = \text{span}\{\varphi_1, \cdots, \varphi_{m_2}\}$, where the ϕ's and φ's are bases of cubic B-splines for the two spaces.

With g in the form $g(x, y) = \sum_{p=1}^{m_1} \sum_{q=1}^{m_2} c_{p,q} \varphi_q(y) \phi_p(x)$, the above interpolation conditions lead to a set of equations

$$\sum_{p=1}^{m_1} \sum_{q=1}^{m_2} c_{p,q} \varphi_q(y_j) \phi_p(x_i) = f_{ij} \tag{12}$$

for all $i = 1, \cdots, m_1$ and $j = 1, \cdots, m_2$. This double sum can be split into two sets of simple sums

$$\sum_{p=1}^{m_1} d_{p,j} \phi_p(x_i) = f_{ij} \tag{13}$$

$$\sum_{q=1}^{m_2} c_{p,q} \varphi_q(y_j) = d_{p,j}. \tag{14}$$

We can interpret (13) and (14) as follows: First, interpolate in the x-direction by determining the piecewise cubic spline curves X_j interpolating the data f_{ij}. And then make a piecewise cubic surface by filling in the space between these curves. Obviously, this process is symmetric in x and y directions.

We can denote the formula (13) in matrix form as

$$\begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_{m_1}(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_{m_1}(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(x_{m_1}) & \phi_2(x_{m_1}) & \cdots & \phi_{m_1}(x_{m_1}) \end{pmatrix} \begin{pmatrix} d_{1,j} \\ d_{2,j} \\ \vdots \\ d_{m_1,j} \end{pmatrix} = \begin{pmatrix} f(x_1, y_j) \\ f(x_2, y_j) \\ \vdots \\ f(x_{m_1}, y_j) \end{pmatrix} \tag{15}$$

for all $j=1,2,\cdots,m_2$.

Similarly, the formula (14) can be written as

$$\begin{pmatrix} \varphi_1(y_1) & \varphi_2(y_1) & \cdots & \varphi_{m_2}(y_1) \\ \varphi_1(y_2) & \varphi_2(y_2) & \cdots & \varphi_{m_2}(y_2) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1(y_{m_2}) & \varphi_2(y_{m_2}) & \cdots & \varphi_{m_2}(y_{m_2}) \end{pmatrix} \begin{pmatrix} c_{i,1} \\ c_{i,2} \\ \vdots \\ c_{i,m_2} \end{pmatrix} = \begin{pmatrix} d_{i,1} \\ d_{i,2} \\ \vdots \\ d_{i,m_2} \end{pmatrix} \tag{16}$$

for all $i=1,2,\cdots,m_1$.

After solving the linear systems (15) and (16), we can determine the control points $c_{i,j}$ $(i=1,2,\cdots,m_1, j=1,2,\cdots,m_2)$. Substituting the control points $c_{i,j}$ into $g(x,y)=\sum_{p=1}^{m_1}\sum_{q=1}^{m_2} c_{p,q}\varphi_q(y)\phi_p(x)$, we find a piecewise cubic spline surface $g(x,y)$ such that $g(x_i,y_j)=f_{ij}$ for $i=1,2,\cdots,m_1$ and $j=1,2,\cdots,m_2$. With this interpolating cubic spline surface $g(x,y)$, we can determine the approximate volume in the following way.

$$\text{Volume} \simeq \sum_{p=1}^{m_1}\sum_{q=1}^{m_2} c_{p,q} \int_c^d \varphi_q(y)\,dy \int_a^b \phi_p(x)\,dx. \tag{17}$$

4. Some computational results

We use Maple software to implement our proposed algorithm. We test three examples with several cases given in Yoo, Lee and Mun (2002).

For each example, we consider the following three cases:

Case1 : With equal intervals [1, 21, 41, 61, 81, 101, 121] in the x-direction, but with unequal intervals [1, 26, 36, 66, 81, 91] in the y-direction.

Case2 : With both unequal intervals [1, 16, 46, 56, 91, 101, 121] in the x-direction and [1, 19, 37, 55, 73, 91] in the y-direction.

Case3 : With both unequal intervals [1, 16, 46, 56, 91, 101, 121] in the x-direction and [1, 26, 36, 66, 81, 91] in the y-direction.

The first example is $f(x,y) = \frac{y^2}{1000} + \frac{xy^2}{10000} + \frac{x^2}{1000}$ for $1 \le x \le 121$ and $1 \le y \le 91$.

We use the exact value with $\int_1^{91}\int_1^{121} f(x,y)\,dx\,dy \simeq 267,160.68(m^3)$.

Here we can summarize computational results with the natural and free boundary conditions.

Table 1: Results and Errors of $f(x,y) = \frac{y^2}{1000} + \frac{xy^2}{10000} + \frac{x^2}{1000}$

	Exact Volume	With Free Boundary Cond.		With Natural Boundary Cond.	
		Approximate Volume	Error	Approximate Volume	Error
Case 1	267,160.68	267,160.68	0.0	268,039.55	0.3
Case 2	267,160.68	267,160.68	0.0	267,681.53	0.2
Case 3	267,160.68	267,160.68	0.0	268,011.72	0.3

The second example is $f(x,y) = \frac{20 + y\sqrt{y}}{10\sqrt{x}}$ for $1 \le x \le 121$ and $1 \le y \le 91$.

We use the exact value with $\int_1^{91}\int_1^{121} f(x,y)\,dx\,dy \simeq 66,795.76421(m^3)$.

Here we can summarize computational results with the natural and free boundary conditions.

Table 2: Results and Errors of $f(x,y) = \frac{20 + y\sqrt{y}}{10\sqrt{x}}$

	Exact Volume	With Free Boundary Cond.		With Natural Boundary Cond.	
		Approximate Volume	Error	Approximate Volume	Error
Case 1	66,795.76	76,187.37	14.1	79,013.37	18.3
Case 2	66,795.76	68,972.52	3.3	68,725.17	2.9
Case 3	66,795.76	68,994.13	3.3	68,881.02	3.1

The third example is $f(x,y)=50\,e^{-\{(\frac{x-60}{40})^2+(\frac{y-45}{30})^2\}}$. for $1\le x\le 121$ and $1\le y\le 91$.

We use the exact value with $\int_1^{91}\int_1^{121} f(x,y)\,dx\,dy \simeq 175877.6457(m^3)$.

Here we can summarize computational results with the natural and free boundary conditions.

Table 3: Results and Errors of $f(x,y)=50\,e^{-\{(\frac{x-60}{40})^2+(\frac{y-45}{30})^2\}}$

	Exact Volume	With Free Boundary Cond.		With Natural Boundary Cond.	
		Approximate Volume	Error	Approximate Volume	Error
Case 1	175,877.65	170,166.13	3.3	176,734.67	0.5
Case 2	175,877.65	173,645.05	1.3	176,747.92	0.5
Case 3	175,877.65	170,120.72	3.3	176,114.72	0.1

Algorithm:

1. Read data: $f(x_i, y_j)$ for $i=1,2,\cdots,m_1$, $j=1,2,\cdots,m_2$.
2. Define the knot vectors for the x and y directions.
   For example, use $(x_1, x_1, x_1, x_1, x_3, x_4, \cdots, x_{m_1-2}, x_{m_1}, x_{m_1}, x_{m_1}, x_{m_1})$ and
   $(y_1, y_1, y_1, y_1, y_3, y_4, \cdots, y_{m_2-2}, y_{m_2}, y_{m_2}, y_{m_2}, y_{m_2})$ for the free boundary conditions.
3. Construct the cubic B-spline by using the formula (10).
4. Solve the formula (13) and then solve the formula (14) to get $c_{i,j}$.
5. Construct the interpolating cubic spline surface with the formula (12).
6. Calculate the approximate volume with the formula (17).

5. Conclusions

We obtain similar accuracy for several examples and cases when we use the interpolating cubic spline surface with the natural and free boundary conditions. We can not say that one method is better than the other method. It only depends on the problem and the case, but the method with free boundary conditions is stabler than the method with the natural boundary conditions in the sense of that the variation of errors is small for one with the free boundary conditions.

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