Reliability of Phased Mission Systems of where Phase Durations are Random Variables

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Abstract

Reliability of multi-phased mission system is represented where redundant components are repairable. Failures and repairs of components follow Markovian property.

Under some constraints, 4 models are available. Two models are represented here. The solutions are obtained as recursive equations using Markov model and eigenvalue system.

1. INTRODUCTION

A phased mission system is defined as "a system where the mission consists of phased sub-missions and whose relevant configuration changes during time periods (phases)." As systems increase in complexity and automation, phased mission analysis is being recognized as the appropriate reliability analysis method [Alam 86]. Fig. 1 shows an example of 3-phased mission system which consists of 3 components, where some components are inactive in some phases.

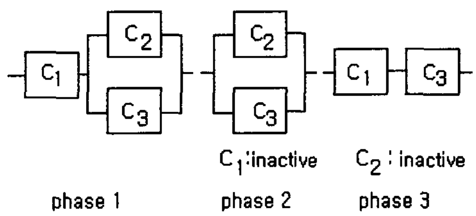

Fig. 1. An Example : 3-phased mission system of 3-components

In past works, there are two classes about multi-phased mission system's reliability; 1) repairable and 2) non-repairable.

The exact reliability of phased mission for non-repairable system is studied by [Esary 75], where the main ideas are; 1) equivalent transformation into single phase, and 2) cut cancellation, that is, a minimal cut in a phase is canceled from the list of minimal cut sets if it contains a minimal cut set of a later phase. [Burdick 77] and [Veatch 86] proposed approximation techniques of non-repairable system. Another exact procedure using Markov model is presented by [Vujosevic 85] for a restricted class of non-repairable components. [Park 90] studied the reliability apportionment problem subject to multiple resources constraints for non-repairable phased mission system.

If a system is repairable, generally it is not simple to obtain the exact reliability. Little methodology has previously been developed to solve general repairable phased mission problems systematically. In the case a system is represented by a set of multi-states according to each component's binary status and transitions among states has Markovian property, [Alam 86] studied this Markov model; a sequential computation of single-phase systems with appropriate initial conditions. [Montague 80] calculated the mean number of failures of a phased mission system when the system is repairable or non-repairable using minimal cut set and minimal path set.

In this paper we assume failure and repair times of components are exponentially distributed and the system has on-board repair facility, ie., redundant components are repairable as long as the system is in up states only. Under these assumptions, which are same as in Alam's work, the system can be described as continuous-time Markov model with appropriate definition of states.

We treat two cases; (1) preliminarily, the phase durations are deterministic. A systematic approach based on Alam's work is reviewed. Then the computational reduced set model is proposed. A programmable solution procedure is developed, introducing eigenvalue system. (2) the phase durations are random variables. 4 models are possible under time constraints. 1) There is no mission time requirements. 2) The full mission must be finished in a time limit. 3) Mission phase change times must be finished in some time limits. 4) each submission duration must be finished in a time limits. We will treat the case 1 and the model 1) of case 2.

2. PRELIMINARY : MISSION RELIABILITY FOR DETERMINISTIC PHASE DURATIONS

2.1 Model Formulation

Notations

t_k : duration of phase k, $1 \le k \le H$

$T_k = \sum_{j=1}^{k} t_j$: mission phase change time (MPCT), where $T_0 = 0$

$\mathbf{s} = [x_j]$: state vector where $x_j = 1$ if component j is good and 0 otherwise, $1 \le j \le c$, where c is the number of components

ϕ_{ki} : system status indicator of state i being up or down in phase k

$\mathbf{p}_k(u) = [p_{ki}(u)]$: probability vector (row type) of phase k where $p_{ki}(u)$ is the probability of being in state i at the elapsed time u from T_{k-1}, $0 \le i \le 2^c - 1$

$A_k = [a_{ij}^{(k)}]$: 2^c dimensional transition rate matrix (TRM) of phase k, $0 \le i,\ j \le 2^c - 1$.

We assume phase durations follows to exponential distributions. There are 2^c possible states according to the status of c components. Let the state identification numbers $i = 0, 1, \cdots, 2^c - 1$. The corresponding state vector $\mathbf{s}_i$ for state i can be obtained by the binary number of i.

Let a system status indicator;

$$\phi_{ki} = \begin{cases} 1, & \text{if } i \text{ is up} \\ 0, & ow \end{cases}$$

Table 1 shows the state description for the example of Fig. 1.

Table 1. states description and state space of phases

state id. number (i)	state vector (x_1, x_2, x_3)	system status indicator		
		ϕ_{1i}	ϕ_{2i}	ϕ_{3i}
7	1 1 1	1	1	1
6	1 1 0	1	1	0
5	1 0 1	1	1	1
4	1 0 0	0	0	0
3	0 1 1	0	1	0
2	0 1 0	0	1	0
1	0 0 1	0	1	0
0	0 0 0	0	0	0

Consider a stochastic process $\{X_k(u), u \in [0, t_k]\}$, where u is the elapsed time from the starting time of phase k (T_k), and an observation $X_k(u)$ is the one of 2^c states at u. This process has time homogeneous Markovian property [Ross 83]. However it is not time homogeneous during whole mission time [0,T] since the configuration is changed at each mission phase change times, ie., transition rates are changed at each phase.

Hence continuous time Markov model can be applied to obtain state probabilities for each phase taking into account the appropriate initial condition [Alam 86].

Let a matrix A_k be the transition rate matrix (TRM) for phase k such that;

$$\begin{cases} a_{ij}^{(k)} : \text{transition rate from state i to j, i} \neq \text{j}, \\ a_{ii(k)} = - \sum_{j \neq i} a_{ij}^{(k)}, \end{cases}$$

where $i,\ j\ =\ 0,\ 1,\ \cdots,\ 2^c-1$. The entries consist of failure rates and repair rates of components.

It is convenient to use a base TRM A to obtain A_k where all transition rates are listed in A assuming failed components are repairable even if the system is down. 'A' is obtained systematically using operations of binary number. Fig.2. shows the base TRM A for the example of Fig.1.

Then, A_k is obtained from A as follows;

$$a_{ij}^{(k)} = \begin{cases} a_{ij}, & \text{if } \phi_{ki}=0 \\ 0, & ow. \end{cases}$$

There are no repair from a down state to another one, and the failure from that is no meaningful.

$$\begin{pmatrix} -\Sigma_0 & \mu_3 & \mu_2 & \cdot & \mu_1 & \cdot & \cdot & \cdot \\ \lambda_3 & -\Sigma_1 & \cdot & \mu_2 & \cdot & \mu_1 & \cdot & \cdot \\ \lambda_2 & \cdot & -\Sigma_2 & \mu_3 & \cdot & \cdot & \mu_1 & \cdot \\ \cdot & \lambda_2 & \lambda_3 & -\Sigma_3 & \cdot & \cdot & \cdot & \mu_1 \\ \lambda_1 & \cdot & \cdot & \cdot & -\Sigma_4 & \mu_3 & \mu_2 & \cdot \\ \cdot & \lambda_1 & \cdot & \cdot & \lambda_3 & -\Sigma_5 & \cdot & \mu_2 \\ \cdot & \cdot & \lambda_1 & \cdot & \lambda_2 & \cdot & -\Sigma_6 & \mu_3 \\ \cdot & \cdot & \cdot & \lambda_1 & \cdot & \lambda_2 & \lambda_3 & -\Sigma_7 \end{pmatrix}$$

Σ_i = row sum except (i,i) entry

Fig.2. Base TRM A of the example

Let $\mathbf{p}_k(u)$ be the states probability row vector $[p_{k0}(u), \cdots, p_{kn}(u)]$ of phase k, where $p_{ki}(u) = \Pr\{X_k(u) = i\}$. The corresponding differential equations to get $\mathbf{p}_k(u)$ are;

$$\frac{d\ \mathbf{p}_k(u)}{du} = \mathbf{p}_k(u) A_k, \quad u \in [0, t_k], 1 \le k \le H. \tag{1}$$

It is known as Kolmogorov's forward equations [Ross 83].

The solution of (1) is

$$\mathbf{p}_k(u) = \mathbf{p}_k(0) e^{A_k u}, \tag{2}$$

where the matrix exponential $e^{A_k u}$ is defined by the matrix series as follows,

$$e^{A_k u} = \sum_{j=0}^{\infty} \frac{u^j [A_k]^j}{j!}.$$

Since the $a_{ii}^{(k)}$ are bounded and state space is finite, this matrix series converges for all values of u [Dyer 89] [Ross 83].

For phase 1, the state probability at the end of phase 1 is

$$\mathbf{p}_1(t_1) = \mathbf{p}_1(0) e^{A_1 t_1}.$$

The final condition of phase 1 is not the initial condition of phase 2 directly. Since system failure during phase 1 gives mission abortion, down states cannot proceed to next phase 2 at T_1. Therefore the initial condition of phase 2 is as follows,

$$p_{2i}(0) = \begin{cases} p_{1i}(t_1), & \text{if } \phi_{1i}=1, \\ 0, & \text{if } \phi_{1i}=0. \end{cases}$$

Generally the initial condition of phase k comes from $\mathbf{p}_{k-1}(t_{k-1})$,

$$p_{ki}(0) = p_{k-1,i}(t_{k-1}) \cdot \phi_{k-1,i} \tag{3}$$

which is a sequentially solvable equation.

Finally, the mission reliability is obtained from $\mathbf{p}_H(t_H)$;

$$R_H = \sum_{j=0}^{n} p_{Hj}(t_H) \cdot \phi_{Hj}. \tag{4}$$

Next we will develop a solution procedure of mission reliability using eigenvalue system to computational compact set of equations.

2.2. Solution of the Reliability with Eigenvalues

More Notations (deleting the phase index k)

Θ : diagonal matrix consisting of eigenvalues θ_i of a matrix A.

$M=[m_{ij}]$: matrix consisting of the eigenvectors, $\mathbf{m}_j$, of A.

$M^{-1}=[w_{ij}]$: inverse of M.

To obtain the solution of (2), some methods are available. The key to the solution is to get matrix exponential e^{Au}. For example 1) an approximation by the matrix series directly, 2) Laplace transformation and algebraic computation, 3) a method using
eigenvalues and eigenvectors, etc.

A method using eigensystem gives theoretically exact solution and it is easily programable with helps of computer and general programs to get the solution of sufficient precision. Further it is appropriate to phased mission reliability, specially to the case of probabilistic phase durations with time constraint.

First, we assume A is "semi-simple", that is, when the eigenvalues of A a) either are distinct, or b) have multiplicities but with corresponding linearly independent eigenvectors, the matrix is called semi-simple [Dief 82] [Pease 65].

There is a eigenvalue problem,

$$A\ \mathbf{m} = \mathbf{m}\theta. \tag{5}$$

As a matrix form,

$$AM = M\Theta.$$

Under assumption of A being semi-simple, M is invertible. Then, by the fact of $A = M\Theta^{-1}M$ and by definition of matrix exponential,

$$e^{Au} = Me^{\Theta u}M^{-1}, \tag{6}$$

where the $e^{\Theta u}$ is a diagonal matrix consisting of $e^{\theta_i u}$. Then its entry is;

$$[e^{Au}]_{ij} = \sum_{\nu=0} m_{i\nu} e^{\theta_\nu u} w_{\nu j}$$

Therefore the solution of final probability should be, by (2) and (3),

$$p_{kj}(t_k) = \begin{cases} \sum_{i=0}^{n} p_{k-1,i}(t_{k-1})\phi_{k-1,i} \sum_{\nu=0}^{n} m_{i\nu}^{(k)} e^{\theta_{k\nu} t_k} w_{\nu j}^{(k)} \\ \sum_{\nu=0}^{n} m_{1\nu}^{(k)} e^{\theta_{1\nu} t_1} w_{\nu j}^{(1)} \end{cases}$$

Therefore the mission reliability R, (4), is obtained by applying above equations sequentially to computation program.

The program can be made simple using recursive relations. Recursion generally provides no saving in storage, nor it will be faster. But recursive code is more compact, and easier to write and understand.

Really, A_k has some rows of 0 vectors, and has no simplicity generally. Instead, B_k of section 3. is required to be semi-simple.

2.3. Reduced Set Computationally Available

More Notations

$r_k(i)$: re-orderd index of a state i in phase k, $0 \leq i \leq n$

$d_k(I)$: state identification number of reorderd index I in phase k, $1 \leq I \leq n_k$

$\mathbf{q}_k(u) = [q_{kI}(u)]$: n_k dimensional up state probability vector (row type) of phase k at the elapsed time u from T_{k-1}

$B_k = [b_{IJ}^{(k)}]$: n_k dimensional reduced TRM from A_k for phase k.

$b_{IJ}^{(k)} = a_{d_k(I),d_k(J)}$, and $b_{II}^{(k)} = a_{d_k(I),d_k(I)} \neq -\sum_{J \neq I} b_{IJ}^{(k)}$.

Once the system becomes a down state, it never returns to an up state. A down state can be an absorbing state [Dyer 89] or all down states may be collapsed into an absorbing state [Kanderhag 78] [Pages 86]. It is sufficient to retain the entries corresponding to up states in TRM and related differential equations to get reliability. We can set reduced state space and corresponding TRM with up states only except down states. It give computational saving against the model of section 2.1.

Instead of binary-valued indictor ϕ_{ki}, we define $r_k(i)$ and $d_k(I)$ as described in Notations. $d_k(I) = d_{k-1}(J)$ means J and I are different reordered indices of an identical state between adjacent phases k-1 and k, then $J = r_{k-1}(d_k(I))$. For example, $d_3(2) = d_2(3) = 5$ and $r_2(d_3(2)) = r_2(5) = 3$. This fact is meaningful to determine the initial condition of each phase.

Then the reduced TRM B_k for phase k can be obtained by deleting the columns and rows related to down states from A, then by reordering the remainder as described in Notations. Notice the B_k is non-singular [Pages 86] in contrast with A_k.

Let $\mathbf{q}_k(u)$ be the up state probability vector of phase k, whose I-th entry is $q_{kI}(u) = p_{k,d_k(I)}(u)$. And similarly as (3)

$$q_{kI}(0) = \begin{cases} q_{k-1,J}(t_{k-1}), & \text{if } d_k(I) = d_{k-1}(J) \\ 0, & ow \end{cases} \\ = q_{k-1,r_{k-1}(d_k(I))}(t_{k-1}), \text{ for } r_{k-1}(d_k(I)) \neq 0. \tag{8}$$

The solution at $u = t_k$ is final condition of phase k,

$$\mathbf{q}_k(t_k) = \mathbf{q}_k(0) e^{B_k t_k}. \tag{9}$$

And its entry is, by (8),

$$q_{kJ}(t_k) = \sum_{I=1}^{n_k} q_{kI}(0)[e^{B_k t_k}]_{IJ} \\ = \sum_{\{I \mid r_{k-1}(d_k(I)) \neq 0\}} q_{kI}(0)[e^{B_k t_k}]_{IJ} \\ = \sum_{\{I \mid r_{k-1}(d_k(I)) \neq 0\}} q_{k-1,r_{k-1}(d_k(I))}(t_{k-1})[e^{B_k t_k}]_{IJ}. \tag{10}$$

where $[e^{B_k t_k}]_{IJ}$ is an entry of $e^{B_k t_k}$, as described in section 2.2. Specially

$$q_{1J}(t_1) = [e^{B_1 t_1}]_{1J},$$

since $\mathbf{q}_1(0) = (1,0,\ldots,0)$.

The mission reliability is get from final condition of final phase,

$$R_H = \sum_{J=1}^{n_H} q_{HJ}(t_H). \tag{11}$$

4. MISSION RELIABILITY FOR PROBABILISTIC PHASE DURATIONS

4.1. Model 1: Without Mission Time Requirement

More Assumptions

1) The phase duration times are independent random variables whose probability density functions are $f_k(t)$. And moment generating functions of p.d.f. are known.

$$L_k(\theta) = \int_0^\infty e^{\theta t} f_k(t)\,dt.$$

2) There is no requirement for mission times.

The expected mission reliability is

$$\begin{aligned} R &= E[R_H(t_1, \cdots, t_H)] \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty R_H(t_1, \cdots, t_H) \prod_{k=1}^{H} f_k(t_k)dt_H \cdots dt_1. \end{aligned} \tag{12}$$

This integral is separable. Let $r_{kj} = E[q_{kj}(t_k)]$, then it is derived from (10) by independency of phase durations.

Hence,

$$r_{kj} = \begin{cases} \displaystyle\sum_{\substack{J=1 \\ \{I \mid \phi = r_{k-1}(d_k(I))>0\}}}^{n_k} \sum_{\nu=0}^{n} r_{k-1,\phi} \cdot m_{i\nu}^{(k)} w_{\nu j}^{(k)} L_{k\nu}, \\ \displaystyle\sum_{\nu=1}^{n_1} m_{1\nu}^{(k)} w_{\nu j}^{(1)} L_{1\nu}. \end{cases} \tag{13}$$

Then the mission reliability becomes $R = \sum_{J=1}^{n_H} r_{HJ}$.

Particularly, if t_k follows exponential distribution $f_k(t) = a_k e^{-a_k t}$, then

$$L_{k\nu} = \frac{a_k}{a_k - \theta_{k\nu}}. \tag{14}$$

5. CONCLUSION

The Markov approach is useful to obtain the reliability of a system when the failure and repair times of components are exponentially distributed. In this paper, we treat a phased mission system where redundant failed components are repairable as long as system is up status.

Under the assumptions, only a reduced set of the linear differential equations related to up states is sufficient for each phase with appropriate TRM and initial condition. We used eigensystem especially in solution process, because it gives theoretically the exact solution to basic model and useful recursive relations to solve the probabilistic phase duration model.

When phase durations are probabilistic, the reliability is obtained similarly using the definition of expectations. If there exist a constraint that the mission must be completed in a given time allowance, assuming the phase durations are according to exponential distributions, the reliability will be represented as a recursive relational equation with more complexity.

Generally reliability problems of phased mission system are complex. The proposed method gives an exact solution theoretically, and the solution can be obtained practically with the help of computer and general computer programs.

REFERENCES

Alam, M. and U. Al-Saggaf, Quantitative reliability evaluation of repairable phased-mission systems using Markov approach, IEEE Tr. Rel., 35, 1986, 498-503

Burdick, G. et al., Phased mission analysis: a review of new developments and an application, IEEE Tr. Rel., 26, 1977, 43

Deif, A., Advanced Matrix Theory for Scientists and Engineer, Halsted Press, 1982

Dyer, D., Unification of reliability /availability /repairability models for Markov systems, IEEE Tr. Rel., 38, 1989, 246-252

Esary, J. and H. Ziehms, Reliability Analysis of Phased Missions, in Reliability Analysis and Fault Tree Analysis, SIAM, Philadelphia, 1975, 213-236

Kanderhag, L., Eigenvalue approach for computing the reliability of Markov systems, IEEE Tr. Rel., 27, 1978, 337-340

Montague, D. and J. Fussel, A Methodology for calculating the expected number of failures of a system undergoing phase mission, Nuclear Science and Engineering, 74, 1980, 199-209

Pages, A. and M. Gondran, System Reliability Evaluation and Prediction in Engineering, Springer-Verlag, 1986

Park, K.S., Reliability apportionment for phased-mission oriented systems, Reliability Engineering and System Safety, 27, 1990, 357-364

Pease, M., Methods of Matrix Algebra, Academic Press, 1965

Ross, S., Stochastic Processes, John Wiley & Sons, 1983

Veatch, M., Reliability of periodic, coherent, binary systems, IEEE Tr. Rel., 35,

1986, 504 - 507
Vujosevic, M. and D. Meade, Reliability evaluation and optimization of redundant dynamic system, IEEE Tr. Rel., 34, 1985, 171-174