유한 요소법을 이용한 4:1 축소관 유동해석에서 점탄성 구성방정식의 비교

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Practical comparison of differential viscoelastic constitutive equations in finite element modeling of planar 4:1 contraction flow

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Introduction

The numerical simulation of viscoelastic flow around a sharp entrance corner has posed the greatest challenge, because it is formidable difficult to obtain the solution for high Deborah number flow. There are several reasons for the failure of the numerical simulation of the planar 4:1 contraction flow at high Deborah number. It has been known that the numerical scheme plays a significant role in the stability of the convergent solution. Also, the solution of a numerical simulation is very dependent on the constitutive equations. Although the numerical technique is important, the selection of the constitutive equation is more crucial.

In this study, we investigate the high Deborah number flow described by the 8 most popular differential constitutive equations such as the Maxwell, Leonov, Giesekus, FENE-P, Larson, White-Metzner models and the Phan-Thien/Tanner(PTT) model of exponential and linear types. We have employed the discrete elastic viscous stress splitting(DEVSS) and streamline upwinding(SU) methods in order to split the extra stress tensor and also stabilize the convective term in the constitutive equation. All the computational results are examined according to mathematical stability analyses of constitutive equations.

Governing equations

We consider a steady, isothermal, incompressible flow. The continuity equation and the conservation of momentum are given by

$$\underline{\nabla} \cdot \underline{v} = 0 \qquad (1)$$

$$\rho \underline{v} \cdot \underline{\nabla} \underline{v} = -\underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}} + \underline{f} \qquad (2)$$

where $\underline{v}$ is the velocity, ρ is the density, p is the pressure and $\underline{\underline{\tau}}$ is the extra stress tensor. The simplest constitutive model developed from the continuum mechanics is the upper convected Maxwell model:

$$\overset{\nabla}{\underline{\underline{c}}} + \frac{1}{\theta}\left(\underline{\underline{c}} - \underline{\underline{\delta}}\right) = \underline{\underline{0}} \qquad (3)$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + G\,\underline{\underline{c}}$$

$\overset{\nabla}{\underline{\underline{c}}}$ is the upper convected time derivative of the tensor $\underline{\underline{c}}$, $\underline{\underline{\sigma}}$ is the total stress tensor and θ is the relaxation time.

The Leonov model based on irreversible thermodynamics is given by

$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{2\theta}\left(\underline{\underline{c}}^{2}+\frac{I_2-I_1}{3}\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}} \tag{4}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\left(\frac{I_1}{3}\right)^{n}\underline{\underline{c}}$$

$$I_1=\mathrm{tr}\,\underline{\underline{c}},\quad I_2=\mathrm{tr}\,\underline{\underline{c}}^{-1},\quad I_3=\det\underline{\underline{c}}=1$$

The simplest Giesekus model is

$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\theta}\left[\alpha\underline{\underline{c}}^{2}+(1-2\alpha)\underline{\underline{c}}-(1-\alpha)\underline{\underline{\delta}}\right]=\underline{\underline{0}} \tag{5}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}}\qquad 0\le\alpha\le 1$$

where α is the positive numerical constant.

The FENE-P constitutive equation is written as

$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\theta}\left(K\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}} \tag{6}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+GK\underline{\underline{c}}$$

$$K=\frac{I_c-3}{I_c-I_1},\quad I_c:const.$$

The Larson model derived from modification of the Doi-Edwards reptation model is

$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\theta}B(I_1)\left(\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}} \tag{7}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+\frac{G\underline{\underline{c}}}{B(I_1)}$$

$$B(I_1)=1+\frac{\xi}{3}(I_1-3),\quad 0\le\xi\le 1$$

The upper convected Phan-Thien/Tanner model of exponential and linear types is given by

$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\theta}\exp[\varepsilon(I_1-3)]\left(\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}} \tag{8}$$

or $$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\theta}[1+\varepsilon(I_1-3)]\left(\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}}$$

The White-Metzner model introduced by modification of an upper convected Maxwell model is

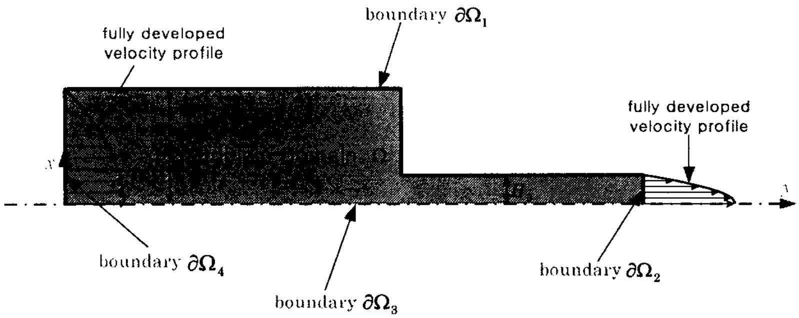
$$\overset{\nabla}{\underline{\underline{c}}}+\frac{1}{\tau(\dot{\gamma})}\left(\underline{\underline{c}}-\underline{\underline{\delta}}\right)=\underline{\underline{0}} \tag{9}$$

$$\underline{\underline{\sigma}}=-p\underline{\underline{\delta}}+G\underline{\underline{c}}$$

where, $\dot{\gamma}=\sqrt{2\underline{\underline{e}}:\underline{\underline{e}}}$ and $\tau(\dot{\gamma})=K_M\dot{\gamma}^{n-1}$.

Numerical scheme

We investigate the planar 4:1 sharp-corner contraction flow with the boundary condition shown in Fig.1.



$\partial\Omega_1$: essential boundary condition for velocity ($\underline{v} = \underline{0}$)

$\partial\Omega_2$: essential boundary condition for velocity or open boundary condition

$\partial\Omega_3$: symmetry boundary condition

$\partial\Omega_4$: essential boundary condition for velocity and $\underline{\underline{c}}$

Fig.1. Boundary conditions of the 4:1 contraction flow problem

The streamline-upwinding scheme and the discrete elastic viscous stress splitting are employed in the numerical solution of viscoelastic flows.
The FE formulation of the Leonov model is expressed as

$$\left\langle \phi_i ;\ \overset{\nabla}{\underline{\underline{c}}} + \frac{1}{2\theta}\left(\underline{\underline{c}}^2 + \frac{I_2 - I_1}{3}\underline{\underline{c}} - \underline{\underline{\delta}}\right)\right\rangle = \underline{\underline{0}}$$

$$\langle \phi_i ;\ \rho \underline{v}\cdot\underline{\nabla v}\rangle = \left\langle \phi_i ;\ \nabla\cdot\left(-p\underline{\underline{\delta}} + G\left(\frac{I_1}{3}\right)^n \underline{\underline{c}} + 2\eta_0\left(\underline{\underline{e}} - \underline{\underline{\bar{e}}}\right)\right) + f_i\right\rangle$$

$$\langle \phi_i ;\ \underline{\nabla}\cdot\underline{v}\rangle = 0$$

$$\left\langle \phi_i ;\ \left(\underline{\underline{e}} - \underline{\underline{\bar{e}}}\right)\right\rangle = 0$$

The symbol $\langle\ ;\ \rangle$ denotes the integration over the domain.

Results and Discussion

We obtained the convergent solution of the 8 differential constitutive equations. The upper limit values of the Deborah number and the mathematical stability characteristics of viscoelastic constitutive equations are shown in Table 1.

Differential Models	**Type of Instability**	**The upper limit of the De**
Upper convected Maxwell	Dissipative unstable	6.67
Leonov CE under Specified Stability constraints	Globally Hadamard and dissipative **stable**	130
Giesekus	Dissipative **stable** $(0 \le \alpha \le 1/2)$	200
	Dissipative unstable $(1/2 \le \alpha \le 1)$	0.19
FENE	Globally Hadamard and dissipative **stable**	5

Larson differential	Hadamard and dissipative unstable	1
White-Metzner Viscosity truncation at shear rate 1 n=0.75	Hadamard and dissipative unstable	10.67
Upper convected Phan-Thien/Tanner (linear)	Hadamard **stable**; Dissipative **stability** Depends on dissipative terms	10
Upper convected Phan-Thien/Tanner (exponential)		17.3

Table 1. The upper limit of the Deborah number and the mathematical stability characteristics (Kwon, 1995)

$De \equiv U\theta / L$ is the Deborah number, where U is average downstream velocity, θ is relaxation time and L is the half width of downstream channel.

It is verified that mathematically well-posed constitutive equations proven from the stability analysis yield convergent numerical solutions in higher Deborah number flows. As a result, solutions for relatively high Deborah number flow can be obtained when one employs the Leonov, the PTT or the Giesekus constitutive equation as a viscoelastic field equation. By examining the region of solutions with loss of evolution, the close relationship between numerical convergence and mathematical stability of model equations is clearly demonstrated.

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