Oscillatory modes generated by Hopf bifurcations in coupled four oscillators

Hiroyuki Kitajima† and Hiroshi Kawakami‡

†Dept. of Reliability-based Information Systems Engineering, Kagawa University
‡Dept. of Electrical and Electronic Engineering, The University of Tokushima
†2217-20 Hayashi, Takamatsu, 761-0396 JAPAN, ‡2-1 Minami-josanjima Tokushima, 770-8506 JAPAN
†Tel: +81-87-864-2226, †Fax: +81-87-864-2262
E-mail: †kitaji@eng.kagawa-u.ac.jp ‡kawakami@ee.tokushima-u.ac.jp

Abstract: We examine the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the four-coupled oscillator system. The Hopf bifurcations of the equilibrium points and the generated oscillatory modes are classified. By numerical bifurcation analysis we observe various interesting synchronized *states caused by symmetry-breaking bifurcations.*

1. Introduction

Systems of coupled oscillators are widely used as models for biological rhythmic oscillations such as human circadian rhythms [1, 2], finger movements [4], animal locomotion [3], swarms of fireflies that flash in synchrony, synchronous firing of cardiac pacemaker cells [5, 6], and so on. Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond to the Hopf bifurcation of an equilibrium point (or the tangent bifurcation of a periodic solution) and the pitchfork bifurcation (or the period-doubling bifurcation) of a periodic solution, respectively. Using group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning with the bifurcations of equilibrium points and periodic solutions [7].

In the study of coupled oscillator system, the four-coupled oscillator system is one of the most interesting system, because there exists an irregular degenerate oscillatory mode (or an independent pair of anti-phase mode) [8, 9] when the equation of the single oscillator is invariant under inversion of state variables.

Mishima and Kawakami studied the oscillatory modes generated by the Hopf bifurcations of the origin (equilibrium point) in several systems of coupled four BVP (Bonhöffer-van der Pol) oscillators [10]. However, they only considered the Hopf bifurcation of the origin, because only the Hopf bifurcation of the origin is supercritical. Tsumoto *et al.* investigated bifurcations of the Modified BVP (MBVP) equation [11]. In the MVBP system, the supercritical Hopf bifurcation of non-origin equilibrium points occurs.

In this paper, we examine the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the four-coupled oscillator system . The Hopf bifurcations of the equilibrium points and the generated oscillatory modes are classified. By numerical bifurcation analysis we observe various interesting synchronized states caused by symmetry-breaking bifurcations.

2. Method of Analysis

We consider the coupled MBVP oscillator system shown in Fig. 1. The circuit equation is described as

$$\begin{aligned} L_1\frac{di_{k1}}{dt} &= -R_1 i_{k1} - v_k \\ L_2\frac{di_{k2}}{dt} &= -R_2 i_{k2} - v_k \\ C\frac{dv_k}{dt} &= i_{k1} + i_{k2} - g(v_k) \\ &\quad -G_a(2v_k - v_{k+1} - v_{k-1}) - G_b(v_k - v_{k+2}) \end{aligned} \tag{1}$$

$$(k = 1, \cdots, 4,\ v_0 \equiv v_4,\ v_5 \equiv v_1,\ v_6 \equiv v_2),$$

where the nonlinear conductance $g(v_k)$ is assumed to be

$$g(v_k) = -v_k + \frac{1}{3}v_k^3. \tag{2}$$

The values of system parameters are fixed as [11]

$$L_1^{-1} = 0.2,\ L_2^{-1} = 0.06,\ R_1 = 4.0,\ R_2 = 2.1,\ C^{-1} = 3.0. \tag{3}$$

The Jacobi matrix of Eq. (1) is described by

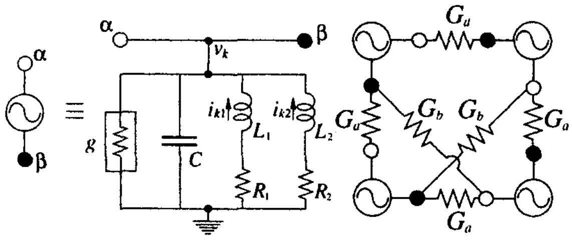
$$DF = \begin{bmatrix} X_0 & X_1 & X_2 & X_1 \\ X_1 & X_0 & X_1 & X_2 \\ X_2 & X_1 & X_0 & X_1 \\ X_1 & X_2 & X_1 & X_0 \end{bmatrix}. \tag{4}$$

Each block is given by

$$X_0 = \begin{bmatrix} -R_1L_1^{-1} & 0 & -L_1^{-1} \\ 0 & -R_2L_2^{-1} & -L_2^{-1} \\ C^{-1} & C^{-1} & C^{-1}(1 - v_{*k}^2) + 2d_a + d_b \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d_a \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d_b \end{bmatrix} \tag{5}$$

where v_{*k} is a coordinate of an equilibrium point, $d_a = -C^{-1}G_a$ and $d_b = -C^{-1}C_b$. Using orthogonal matrix



(a) (b)

Figure 1. MBVP circuit (a) and coupled system (b).

given by

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2}\,I & I & O & 1/\sqrt{2}\,I \\ 1/\sqrt{2}\,I & O & I & -1/\sqrt{2}\,I \\ 1/\sqrt{2}\,I & -I & O & 1/\sqrt{2}\,I \\ 1/\sqrt{2}\,I & O & -I & -1/\sqrt{2}\,I \end{bmatrix} \tag{6}$$

we diagonalize the Jacobian matrix (4) as:

$$Q^{-1} \cdot DF \cdot Q = \begin{bmatrix} Y_0 & O & O & O \\ O & Y_1 & O & O \\ O & O & Y_1 & O \\ O & O & O & Y_2 \end{bmatrix} \tag{7}$$

where

$$Y_0 = X_0 + 2X_1 + X_2, \tag{8}$$

$$Y_1 = X_0 - X_2, \tag{9}$$

$$Y_2 = X_0 - 2X_1 + X_2. \tag{10}$$

In the next section we classify the oscillatory modes generated by the Hopf bifurcation in each block Y_l (l = 1, 2, 3) in Eq. (7).

3. Results

In Eq. (1) there exists four equilibrium points satisfying $v_{*1}^2 = v_{*2}^2 = v_{*3}^2 = v_{*4}^2$; those are $(v_{*1}, v_{*2}, v_{*3}, v_{*4})$ = (0, 0, 0, 0), (a_1, a_1, a_1, a_1), $(a_2, -a_2, a_2, -a_2)$ and $(a_3, a_3, -a_3, -a_3)$. Here we study the Hopf bifurcation of equilibrium points $(a_2, -a_2, a_2, -a_2)$ and $(a_3, a_3, -a_3, -a_3)$ named type 1 and 2, respectively.

Figure 2 shows a bifurcation diagram of type 1 and 2 equilibrium points. In this diagram, the line ${}_mh_l$ indicates the Hopf bifurcation set of the type m (1 or 2) equilibrium point and of the block Y_l. In the shaded region ☐ and ▬, the stable type 1 and type 2 equilibrium point exists, respectively. Thus, supercritical Hopf bifurcation sets are ${}_1h_1$, ${}_1h_2$, ${}_2h_1$ and ${}_2h_2$. In Tab. 1 we summarize oscillatory modes generated by Hopf bifurcations shown in Fig. 2.

The closed circle in Fig. 2 indicates the Hopf-Hopf codimension -two bifurcation point [12]. From this point the Neimark-Sacker bifurcation sets of periodic solutions generated by each Hopf bifurcation appear.

We show in Figs. 3 and 4 waveforms of a pair of in-phase generated by ${}_1h_2$ and a pair of anti-phase generated by ${}_2h_2$, respectively. By changing the values of coupling coefficients d_a and d_b, symmetry-breaking bifurcations (the pitchfork bifurcations) occur and the solutions of Figs. 3 and 4 bifurcate to those of Figs. 5 and 6, respectively.

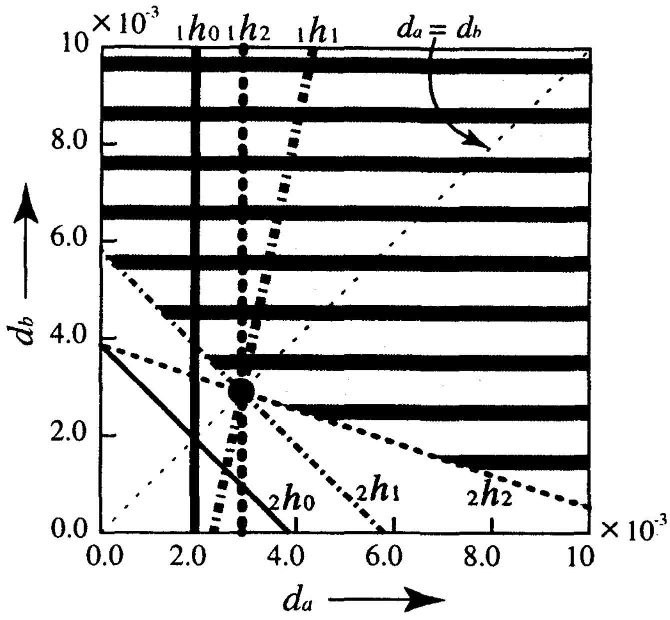


Figure 2. Bifurcation diagram of equilibrium points observed in Eq. (1).

After the bifurcation the amplitude of oscillator ① and ② (③ and ④) is different, but the in-phase synchronized state between ① and ③ (② and ④) is kept (see Fig. 5). It is quite interesting that the single MBVP is not a hard oscillator, but in the coupled system the synchronized state with different amplitude is observed. In Fig. 6 the oscillators ① and ② (③ and ④) also have different amplitude, but the ② and ③ (① and ④) are synchronized at anti-phase (waveform of ② times −1 equals that of ③). By decreasing the value of d_a symmetry-breaking bifurcation occur again, and only oscillator ④ is stopped and the others seem to have similar amplitude (see Fig. 7). The solutions of Figs. 5 and 7 meet the Neimark-Sacker bifurcation, because in the neighborhood there exists the Hopf-Hopf codimension-two bifurcation.

We show an interesting oscillatory mode in Fig. 8. Two oscillators (① and ③) are synchronized at anti-phase and the other two (② and ④) are non-oscillatory. By changing the values of coupling coefficient this solution becomes a pair of almost in-phase and a pair of almost anti-phase (see Fig. 9).

4. Concluding Remarks

We have investigated the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the system of coupled four MBVP oscillators. The Hopf bifurcations of the equilibrium points and the generated oscillatory modes are classified. Moreover, by numeri-

Table 1. Classification of oscillatory modes.

Y_l	Hopf	Type 1	Hopf	Type 2
Y_0	${}_1h_0$	a pair of in-phase	${}_2h_0$	a pair of in-phase
Y_1	${}_1h_1$	a pair of anti-phase	${}_2h_1$	an independent pair of anti-phase
Y_2	${}_1h_2$	a pair of in-phase	${}_2h_2$	a pair of anti-phase

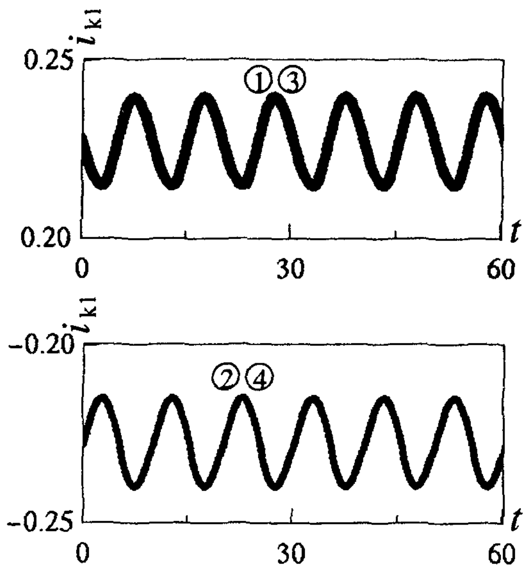


Figure 3. Periodic solution generated by ${}_1h_2$ observed in Eq. (1) with $d_a = 0.0028$ and $d_b = -0.01$. Circled numbers indicate k in Eq. (1).

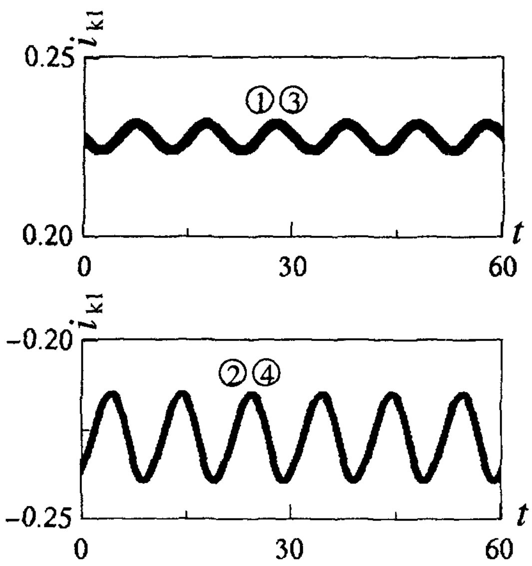


Figure 5. Periodic solution generated by the pitchfork bifurcation of Fig. 3. $d_a = 0.0024$. $d_b = -0.01$.

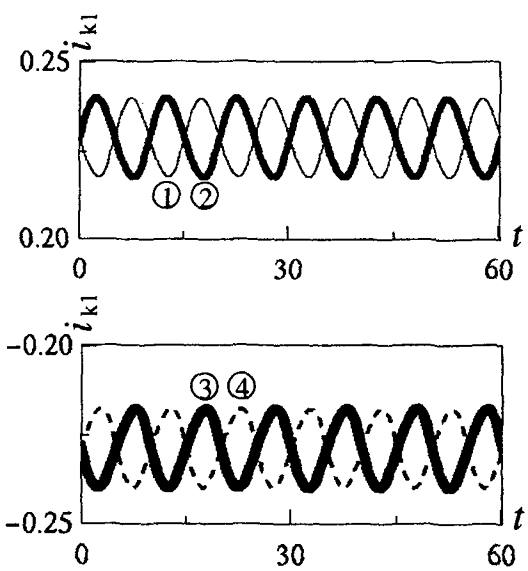


Figure 4. Periodic solution generated by ${}_2h_2$. $d_a = 0.01$. $d_b = 0.0004$.

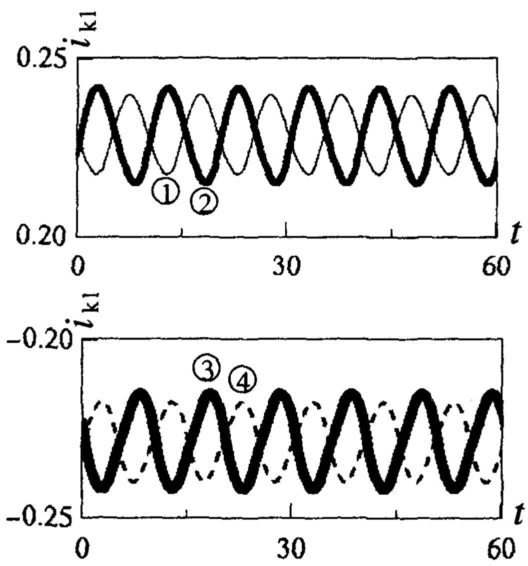


Figure 6. Periodic solution generated by the pitchfork bifurcation of Fig. 4. $d_a = 0.01$. $d_b = 0.0003$.

cal bifurcation analysis we observed various interesting synchronized states caused by symmetry-breaking bifurcations.

Considering the associative memory model for storing patterns as oscillatory states [13], this system has the advantage of many oscillatory modes.

Acknowledgment

This work was partially supported by the Grant-in-Aid for Young Scientists (B) (No. 14780326) from the Min-

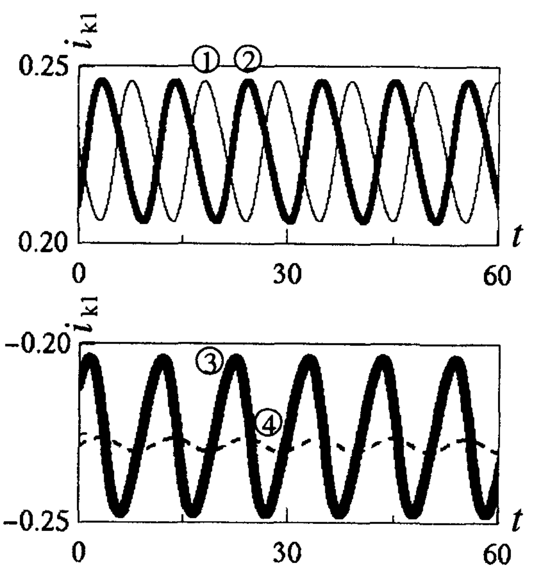


Figure 7. Periodic solution generated by the pitchfork bifurcation of Fig. 6. $d_a = 0.005$. $d_b = 0.0003$.

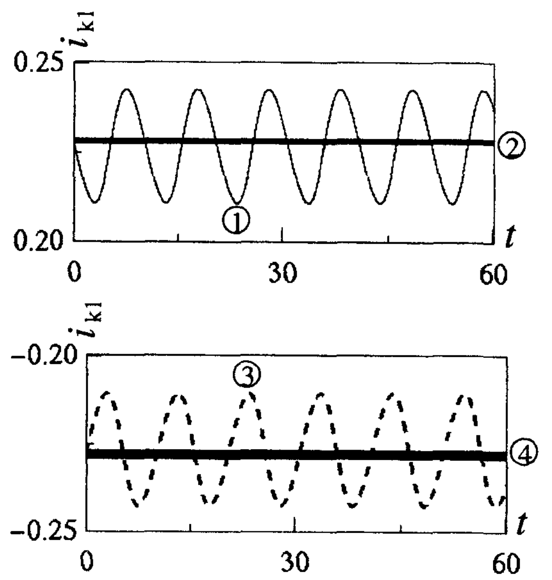


Figure 8. Waveforms of oscillatory modes. with $d_a = d_b = 2.7 \times 10^{-3}$.

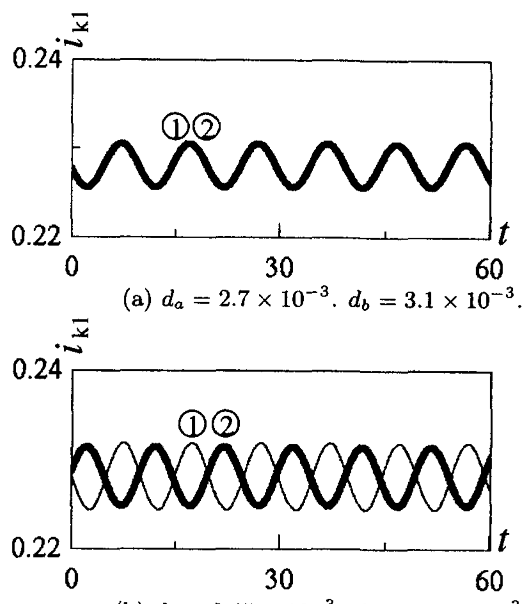


Figure 9. Waveforms of oscillatory modes.

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