Weighted Least-Squares Design and Parallel Implementation of Variable FIR Filters

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Abstract: This paper proposes a weighted least-squares (WLS) method for designing variable one-dimensional (1-D) FIR digital filters with simultaneously variable magnitude and variable non-integer phase-delay responses. First, the coefficients of a variable FIR filter are represented as the two-dimensional (2-D) polynomials of a pair of spectral parameters: one is for tuning the magnitude response, and the other is for varying its non-integer phase-delay response. Then the optimal coefficients of the 2-D polynomials are found by minimizing the total weighted squared error of the variable frequency response. Finally, we show that the resulting variable FIR filter can be implemented in a parallel form, which is suitable for high-speed signal processing.

1 Introduction

Since variable digital filters have been found very useful in various fields, the design and implementation of variable digital filters have received considerable attention recently. Many methods have been developed for designing variable filters with *either* variable magnitude [1, 2] *or* variable fractional-delay (FD) responses [3, 4] only. In applying variable FD filters, one needs to cascade a normal digital filter with a variable FD filter to perform the standard operations because the variable FD filter itself does not possess frequency selectivity, and thus cannot rectify the shape of the overall magnitude response. The cascaded normal frequency-selective digital filter is used to cut off some frequency components and pass through the desired ones. Thus the standard operations require two digital filters; one is for shaping magnitude response, and the other is for delaying digital signals by any time period which does not have to be an integer multiple of the sampling interval. In some applications, both magnitude and FD response are required to be tunable, which compels us to cascade two variable filters; one is for varying magnitude response, and the other is for varying fractional-delay. Therefore, it is strongly desirable to design one single variable filter that can perform the above two variable operations simultaneously.

This paper proposes a WLS method for designing variable 1-D FIR filters whose magnitude and non-integer phase-delay responses can be continuously and independently tuned. First, we assume that the desired variable frequency response is specified by a pair of spectral parameters; one is used for tuning the magnitude response, and the other is for tuning fractional-delay. Then the coefficients of the variable 1-D FIR filter are expressed as the 2-D polynomials of the two spectral parameters. Finally, the optimal coefficients of the 2-D polynomials are determined by minimizing the total weighted squared error of the variable frequency response. We will also show that the resulting variable filter can be implemented in a parallel form that consists of *constant part* and *variable part*. In practical applications, only the variable part needs to be varied for obtaining different (tunable) magnitude and non-integer phase-delay responses. As a result, the obtained parallel-form variable filters are suitable for high-speed signal processing.

2 Design and Implementation

Assume that the ideal variable frequency response is

$$H_I(\omega, \Psi, d) = M_I(\omega, \Psi)e^{j\theta_I(\omega,d)} \tag{1}$$

where ω, $\omega \in [0, \pi]$, is the normalized angular frequency, and Ψ, $\Psi \in [\Psi_{min}, \Psi_{max}]$, is a spectral parameter that specifies the desired variable magnitude response $M_I(\omega, \Psi)$. Also, $\theta_I(\omega, d)$ represents the ideal variable linear phase

$$\theta_I(\omega, d) = -d\omega$$

where d, $d \in [-0.5, 0.5]$, is the desired fractional phase-delay. Obviously, the desired magnitude response $M_I(\omega, \Psi)$ and the desired fractional phase-delay response can be independently varied by using the spectral parameters Ψ and d, respectively. Our objective here is to find the optimal FIR filter

$$H(z, \Psi, d) = \sum_{k=-K/2}^{K/2} a_k(\Psi, d)z^{-k} \tag{2}$$

whose coefficients are the 2-D polynomials

$$a_k(\Psi, d) = \sum_{p=0}^{P}\sum_{q=0}^{Q} b(k, p, q)\Psi^p d^q \tag{3}$$

such that the weighted squared error

$$J_c = \int_0^{\pi}\int_{\Psi_{min}}^{\Psi_{max}}\int_{-0.5}^{0.5} W(\omega, \Psi, d)|e(\omega, \Psi, d)|^2 d_d d_\Psi d_\omega \tag{4}$$

is minimized, where the frequency response error is

$$e(\omega, \Psi, d) = H(\omega, \Psi, d) - H_I(\omega, \Psi, d)$$

and $H(\omega,\Psi,d)$ is the actual variable frequency response of the filter (2). In addition, the $W(\omega,\Psi,d)$ in (4) is a completely separable non-negative weighting function

$$W(\omega,\Psi,d) = W_1(\omega)W_2(\Psi)W_3(d).$$

Substituting (3) into (2) obtains the variable filter

$$H(z,\Psi,d) = \sum_{k=-K/2}^{K/2}\sum_{p=0}^{P}\sum_{q=0}^{Q} b(k,p,q)z^{-k}\Psi^p d^q \quad (5)$$

whose frequency response is

$$H(\omega,\Psi,d) = \sum_{k=-K/2}^{K/2}\sum_{p=0}^{P}\sum_{q=0}^{Q} b(k,p,q)e^{-jk\omega}\Psi^p d^q. \quad (6)$$

Consequently, our objective is to find the optimal coefficients $b(k,p,q)$ such that the total weighted squared error (4) is minimized. From (6) it is known that the number of the total coefficients $b(k,p,q)$ to be determined is $(K+1)(P+1)(Q+1)$. The computational complexity required for finding $b(k,p,q)$ can be reduced by exploiting the coefficient symmetries

$$b(-k,p,q) = (-1)^q \cdot b(k,p,q), \quad k = 0 \sim K/2 \quad (7)$$

which can be proved by using the relation

$$H(\omega,\Psi,-d) = H^*(\omega,\Psi,d).$$

Therefore, the actual variable frequency response can be simplified as

$$H(\omega,\Psi,d) = \sum_{k=0}^{K/2}\sum_{p=0}^{P}\sum_{q=0}^{Q} b(k,p,q)\Omega_k(\omega)\Psi^p d^q \quad (8)$$

where

$$\Omega_k(\omega) = \begin{cases} 1 & \text{if } k=0 \\ e^{-jk\omega} + (-1)^q \cdot e^{jk\omega} & \text{if } k \neq 0. \end{cases} \quad (9)$$

By exploiting the coefficient symmetries (7), we only need to find almost half of the coefficients $b(k,p,q)$, and thus can reduce the computational complexity required in the design.

To minimize J_c in (4), we first sample the parameters

$$\omega \in [0,\pi], \quad \Psi \in [\Psi_{min},\Psi_{max}], \quad d \in [-0.5, 0.5]$$

to obtain the discrete points

$$\begin{aligned} \omega_l &= \frac{(l-1)\pi}{L-1}, & l &= 1 \sim L \\ \Psi_m &= \Psi_{min} + \frac{(\Psi_{max}-\Psi_{min})(m-1)}{M-1}, & m &= 1 \sim M \\ d_n &= -0.5 + \frac{n-1}{N-1}, & n &= 1 \sim N. \end{aligned} \quad (10)$$

Thus we get the corresponding samples as

$$\begin{aligned} H_I(l,m,n) &= M_I(\omega_l,\Psi_m)e^{j\theta_I(\omega_l,d_n)} \\ H(l,m,n) &= \sum_{k=0}^{K/2}\sum_{p=0}^{P}\sum_{q=0}^{Q} b(k,p,q)\Omega_k(\omega_l)\Psi_m^p d_n^q \\ W(l,m,n) &= W_1(\omega_l)W_2(\Psi_m)W_3(d_n). \end{aligned}$$

Then we want to find the coefficients $b(k,p,q)$ by minimizing

$$\begin{aligned} J_d &= \sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} W(l,m,n)\left|H(l,m,n) - H_I(l,m,n)\right|^2 \\ &= \sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} W(l,m,n)\times \\ &\left|\sum_{k=0}^{K/2}\sum_{p=0}^{P}\sum_{q=0}^{Q} b(k,p,q)\Omega_k(\omega_l)\Psi_m^p d_n^q - H_I(l,m,n)\right|^2 \end{aligned} \quad (11)$$

where J_d is the discrete version of the continuous error function J_c in (4). To simplify the design problem formulation, we perform the one-to-one index mappings

$$\begin{aligned} (l,m,n) &\longrightarrow i_1, \quad i_1 = 1,2,\cdots,I_1 \\ (k,p,q) &\longrightarrow i_2, \quad i_2 = 1,2,\cdots,I_2 \end{aligned} \quad (12)$$

where $I_1 = LMN, I_2 = (K/2+1)(P+1)(Q+1)$. Based on the index mappings (12), we can obtain

$$\begin{aligned} b(k,p,q) &\longrightarrow c(i_2) \\ \Omega_k(\omega_l)\Psi_m^p d_n^q &\longrightarrow \Phi(i_1,i_2) \\ H_I(l,m,n) &\longrightarrow h(i_1) \\ W(l,m,n) &\longrightarrow v(i_1). \end{aligned} \quad (13)$$

Thus we can write the error function J_d as

$$\begin{aligned} J_d &= \sum_{i_1=1}^{I_1} v(i_1)\left|\sum_{i_2=1}^{I_2} c(i_2)\Phi(i_1,i_2) - h(i_1)\right|^2 \\ &= \sum_{i_1=1}^{I_1} v(i_1)\left[\sum_{i_2=1}^{I_2} c(i_2)\Phi(i_1,i_2) - h(i_1)\right]\times \\ &\quad \left[\sum_{i_2=1}^{I_2} c(i_2)\Phi^*(i_1,i_2) - h^*(i_1)\right] \end{aligned} \quad (14)$$

where $[\,.\,]^*$ means the complex-conjugate of the complex number $[\,.\,]$. To minimize the error function J_d, we differentiate J_d with respect to the i-th entry of the coefficient vector $\boldsymbol{c}$ and then set the differentiation to zero, which leads to

$$\begin{aligned} &\text{Re}\left[\sum_{i_2=1}^{I_2} c(i_2)\sum_{i_1=1}^{I_1} v(i_1)\Phi(i_1,i)\Phi^*(i_1,i_2)\right] = \\ &\text{Re}\left[\sum_{i_1=1}^{I_1} v(i_1)\Phi(i_1,i)h^*(i_1)\right] \end{aligned} \quad (15)$$

where Re$[\,.\,]$ denotes the real part of $[\,.\,]$. Substituting $i = 1,2,\cdots,I_2$ to the equation (15) obtains

$$\text{Re}\left[\boldsymbol{\Phi}^*\boldsymbol{V}\boldsymbol{\Phi}\right]\boldsymbol{c} = \text{Re}\left[\boldsymbol{\Phi}^*\boldsymbol{V}\boldsymbol{h}\right] \quad (16)$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi(1,1) & \Phi(1,2) & \cdots & \Phi(1,I_2) \\ \Phi(2,1) & \Phi(2,2) & \cdots & \Phi(2,I_2) \\ \vdots & \vdots & \vdots & \vdots \\ \Phi(I_1,1) & \Phi(I_1,2) & \cdots & \Phi(I_1,I_2) \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} v(1) & & & \\ & v(2) & & \\ & & \ddots & \\ & & & v(I_1) \end{bmatrix}$$

$$c = \begin{bmatrix} c(1) \\ c(2) \\ \vdots \\ c(I_2) \end{bmatrix}, \qquad h = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(I_1) \end{bmatrix}.$$

The equation (16) leads to the optimal solution

$$c = \{\mathrm{Re}[\Phi^* V \Phi]\}^{-1} \cdot \mathrm{Re}[\Phi^* V h] \tag{17}$$

but the direct computation (17) is not efficient enough, which can be further modified as follows.

Let the i_1-th column vector of Φ^* be e_{i_1}. Then

$$\Phi^* V \Phi = \sum_{i_1=1}^{I_1} v(i_1) e_{i_1} e_{i_1}^*, \quad \Phi^* V h = \sum_{i_1=1}^{I_1} v(i_1) h(i_1) e_{i_1}. \tag{18}$$

Thus the equation (17) can be re-written as

$$\begin{aligned} c &= \left\{ \mathrm{Re}\left[\sum_{i_1=1}^{I_1} v(i_1) e_{i_1} e_{i_1}^* \right] \right\}^{-1} \cdot \mathrm{Re}\left[\sum_{i_1=1}^{I_1} v(i) h(i_1) e_{i_1} \right] \\ &= \left\{ \sum_{i_1=1}^{I_1} v(i_1) \mathrm{Re}\left[e_{i_1} e_{i_1}^* \right] \right\}^{-1} \cdot \left\{ \sum_{i_1=1}^{I_1} v(i_1) \mathrm{Re}\left[h(i_1) e_{i_1} \right] \right\} \\ &= \Gamma^{-1} \beta \end{aligned} \tag{19}$$

where

$$\Gamma = \sum_{i_1=1}^{I_1} v(i_1) \mathrm{Re}\left[e_{i_1} e_{i_1}^* \right]$$

$$\beta = \sum_{i_1=1}^{I_1} v(i_1) \mathrm{Re}\left[h(i_1) e_{i_1} \right].$$

Since the matrix Γ is positive definite, it can be decomposed as

$$\Gamma = U^t U \tag{20}$$

by using the Cholesky decomposition, where U is an upper triangular matrix, thus

$$\Gamma^{-1} = U^{-1} U^{-t}. \tag{21}$$

Consequently, the optimal vector c can be determined by

$$c = U^{-1}(U^{-t}\beta). \tag{22}$$

The indirect inversion (21) of the matrix Γ is very important for avoiding the ill-conditioning numerical problem when the condition number of Γ is large, thus the final expression (22) provides a numerically stabilized optimal solution.

Once the optimal vector c is obtained, the reverse order of (13) can be applied to determine the optimal coefficients $b(k,p,q)$ as

$$c(i_2) \longrightarrow b(k,p,q). \tag{23}$$

Since the resulting variable FIR filter (5) is noncausal, it cannot be applied in real-time signal procesing. However, a causal one can be easily obtained by just shifting the coefficients $b(k,p,q)$ by K/2 along the k axis, i.e.,

$$\alpha(k,p,q) = b(k - K/2, p, q), \quad k = 0, 1, \cdots, K \tag{24}$$

which results in the final causal variable filter

$$G(z,\Psi,d) = \sum_{k=0}^{K} \sum_{p=0}^{P} \sum_{q=0}^{Q} \alpha(k,p,q) z^{-k} \Psi^p d^q. \tag{25}$$

Note that $G(z,\Psi,d)$ has the same variable magnitude response as $H(z,\Psi,d)$, but its desired variable non-integer phase-delay is

$$D = \frac{K}{2} + d, \qquad d \in [-0.5, 0.5].$$

From (25) we can re-arrange $G(z,\Psi,d)$ as

$$G(z,\Psi,d) = \sum_{p=0}^{P} \sum_{q=0}^{Q} \left[\sum_{k=0}^{K} \alpha(k,p,q) z^{-k} \right] \Psi^p d^q. \tag{26}$$

Letting

$$H_{pq}(z) = \sum_{k=0}^{K} \alpha(k,p,q) z^{-k} \tag{27}$$

we obtain

$$G(z,\Psi,d) = \sum_{p=0}^{P} \sum_{q=0}^{Q} H_{pq}(z) \Psi^p d^q \tag{28}$$

where $H_{pq}(z)$ can be regarded as constant filters, and $\Psi^p d^q$ correspond to the weighting coefficients for $H_{pq}(z)$. Since the weighting coefficients $\Psi^p d^q$ are variable for different values of Ψ and d, thus the variable filter $G(z,\Psi,d)$ consists of *constant part* $H_{pq}(z)$ and *variable part* $\Psi^p d^q$, it can be implemented in the parallel form as Fig. 1. In digital signal processing applications, $H_{oo}(z)$, $H_{o1}(z)$, $\cdots$, $H_{PQ}(z)$ are fixed, and $\Psi^0 d^0$, $\Psi^0 d^1$, $\cdots$, $\Psi^P d^Q$ are varied (variable part). The parallel-form implementation is suitable for high-speed signal processing.

3 Design Example

In this section, we present a numerical example to illustrate the effectiveness of the proposed design method.

[Variable Lowpass Filter]: The desired variable lowpass frequency response

$$H_I(\omega,\Psi,d) = M_I(\omega,\Psi) e^{j\theta_I(\omega,d)}$$

is specified by

$$M_I(\omega,\Psi) = \begin{cases} 1 & 0 \le \omega \le \omega_p \\ \dfrac{\omega_s - \omega}{0.24\pi} & \omega_p \le \omega \le \omega_s \\ 0 & \omega_s \le \omega \le \pi \end{cases}$$

$$\begin{aligned} &\omega_p = 0.26\pi + \Psi, \omega_s = 0.50\pi + \Psi, \Psi \in [-0.16\pi, 0.16\pi] \\ &\theta_I(\omega,d) = -d\omega, \quad d \in [-0.5, 0.5], \quad \omega \in [0,\pi] \end{aligned} \tag{29}$$

where the spectral parameter Ψ controls the passband and the stopband widths, but the transition band width is fixed (0.24π) [1, 2].

Following the proposed design procedures, we first sample the parameters ω, Ψ, and d to get a set of discrete points ω_l, Ψ_m, d_n, where $L = 51, M = 17, N = 11$.

Then the variable filter of order $(K, P, Q) = (30, 4, 4)$ is designed. It should be noted that the filter order should be chosen by designers with the tradeoff between design accuracy and computational complexity. In addition, the weighting function

$$W(\omega, \Psi, d) = W_1(\omega)W_2(\Psi)W_3(d)$$

is carefully selected as

$$W_1(\omega_l) = \begin{cases} 1 & \omega_l \notin [\omega_p, \omega_s] \\ 0 & \omega_l \in [\omega_p, \omega_s] \end{cases}$$
$$W_2(\Psi_m) = [210 \quad 200 \quad 45 \quad 30 \quad 25 \quad 20 \quad 0 \quad 15 \quad 5 \quad 0 \quad 0 \quad 0 \quad 30 \quad 1 \quad 20 \quad 40 \quad 25]$$
$$W_3(d_n) = 1 \quad \text{for all } d_n \tag{30}$$

such that the frequency response errors could be almost uniformly distributed along the ω, Ψ, and d axes. In this case, the maximum and minimum values of the normalized root-mean-squared (RMS) errors of variable magnitude responses are 0.3582% and 0.2289%, respectively. Fig. 2 illustrates the actual variable magnitude response for $d = 0$, and Fig. 3 depicts the passband non-integer phase-delays for $\Psi = 0$, which are considerably flat (constant).

4 Conclusion

We have proposed a closed-form WLS method for designing 1-D FIR digital filters with simultaneously variable magnitude and non-integer phase-delay responses. The resulting variable FIR filters can be implemented in parallel forms, which are suitable for high-speed signal processing. Since the variable FIR filter designed by this method have much more flexibilities than the conventional ones with *either* variable magnitude *or* variable FD responses, the variable FIR filters obtained here are widely useful in the applications where *both* magnitude and non-integer phase-delay responses are required to be tunable.

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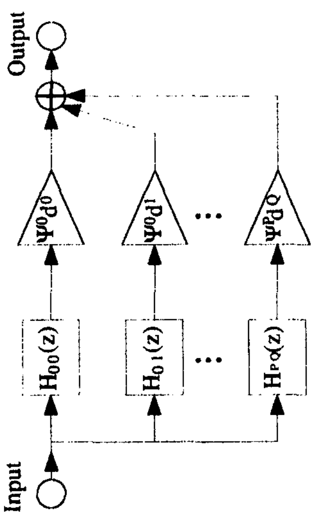

Fig. 1. Parallel Implementation of Variable Filter.

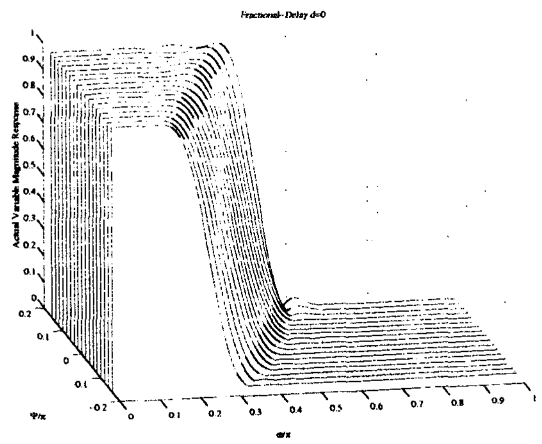

Fig. 2. Variable Magnitude Response for $d = 0$.

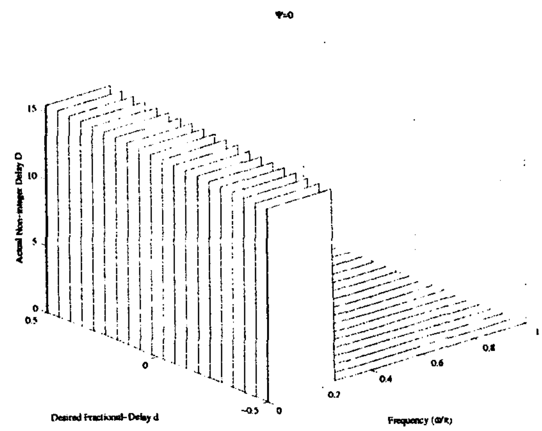

Fig. 3. Variable Non-integer Delay for $\Psi = 0$.